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Dispersion and attenuation of acoustic waves in randomly heterogeneous media

J.O. Parra *, C.L. Hackert, R. Ababou, M.J. Sablik

Instrumentation and Space Research Division, Southwest Research Institute, P.O. Drawer 28510, San Antonio, TX, 78228-0510, USA

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Abstract

We derive the effective displacement relation for acoustic waves in a spatially random heterogeneous one-dimensional medium. This relationship is expressed in terms of parameters σ_R and σ_A which represent the standard deviations of the randomly varying density $\rho(x)$ and the randomly varying Young's modulus $\alpha(x)$, of the medium. In this way, we build the contributions into the total displacement relationship for the spatially random heterogeneous medium and apply this result to determine the dispersion and attenuation of acoustic waves propagating in the random heterogeneous medium. Attenuation and dispersion of waves propagating in media with randomly varying properties has been the subject of much study. Most of this work has neglected the effects of intrinsic dispersion and attenuation in order to concentrate on the effects of the medium inhomogeneities. We demonstrate how intrinsic attenuation may be easily included in the theoretical development, and explore the combined effects of scattering-based and intrinsic attenuation and dispersion on wave propagation. We apply the solution to model interwell acoustic waves propagating in the Kankakee formation at the Buckhorn Test Site, IL. The modeling results show that the strong dispersion in the frequency range of 500–2000 Hz is due to the reservoir heterogeneity. Alternatively, the velocity dispersion for frequencies greater than 2000 Hz corresponds to the intrinsic properties of the reservoir. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

As geophysicists obtain and require more accurate information about acoustic velocities, the dispersion, or frequency dependence of these velocities becomes more important. Dispersion is known to fall into two basic classes: intrinsic, which is based on anelasticity; and scattering, which is based on local wavelength-scale variations in the rock formation. Intrinsic dispersion is a local property of the rock. Scattering dispersion is a property of a neighborhood of rocks, and includes the effects of reflections, refractions, and the law requiring continuity of displacement.

This paper builds on previous one-dimensional and plane wave stochastic random media solutions to provide a more complete theory. Backus (1962) was the first to examine how waves propagate through small scale variations,

* Corresponding author. Tel.: +1-210-522-3284; fax: +1-210-647-4325; E-mail: jparra@swri.edu

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but restricted his work to the zero-frequency limit and did not consider scattering attenuation. O'Doherty and Anstey (1971) took a slightly different approach, examining how multiple reflections from a plane layered medium contribute to dispersion and attenuation. They were the first to introduce the statistical concept of the power spectrum of a stochastic variable as being critical to observed behavior of waves. Burridge et al. (1988, 1989) provided a rigorous proof of the O'Doherty and Anstey result for the case of a particular type of plane layered medium. Nevertheless, by limiting their analysis to plane reflectors, they eliminate the possibility of continuous variability in medium properties. This obstacle was overcome by Shapiro and Zien (1993) and Shapiro et al. (1994) who treated the medium as having a constant mean property value and a stochastic perturbation to this value.

This paper treats the effects of inhomogeneities in the medium as a perturbation to the one-dimensional wave equation. The solution is obtained in the Fourier-transformed coordinates, and so heterogeneities of varying length scale may be included as a perturbation spectrum in the wave number domain. In contrast to an earlier work (Parra et al., 1999), the present derivation is approached in an ensemble average sense to facilitate the inclusion of second-order effects.

The present work has several advantages. It obtains both the frequency dependent scattering dispersion and the frequency dependent scattering attenuation. It is based on a Green's function kernel, and so by replacing the Green's function the theory may be readily extended to additional dimensions and alternate formations. (An extension of this work to two dimensions, incorporating both compressional and shear modes, is in progress.) Like Shapiro et al., this theory employs the spectral density of the random medium, and so may be applied to both discontinuous layered media and continuously variable media in a generalization of O'Doherty and Anstey's results. Finally, this work is unique in incorporating the arbitrary intrinsic attenuation of the medium directly into the theory of the stochastic medium formulation, although Kneib and Shapiro (1995) look at separating scattering attenuation from intrinsic attenuation using synthetic seismograms with a particular intrinsic attenuation mechanism.

In order to develop a vector wave displacement solution associated with the heterogenous medium, we form a relation for the wave displacement in terms of second-order displacements, the variances, and products of standard deviations of the rock physical properties. The second-order terms are associated with the Gaussian random functions $R(x)$ and $A(x)$. The next step in the analysis is to insert the second-order solution in the one-dimensional heterogeneous wave equation, which contains the randomly varying density $\rho(x)$ and randomly varying Young's modulus $\alpha(x)$. These functions are exponential forms of the zero-mean fluctuations $R'(x)$ and $A'(x)$ and standard deviations σ_R and σ_A. The exponential forms of these physical properties are expressed in second-order Taylor expansions and substituted into the heterogeneous wave equation. Thus, the first part of the theory (Appendix A) is devoted to deriving explicit expressions for the second-order forcing functions of the heterogeneous wave equation. The second part of the theory below is devoted to deriving the wave displacement in terms of the varying propagation vector and explicit expressions for the effective wave propagation vector. The last part of the theory is devoted to incorporating the intrinsic attenuation into the theory of the stochastic formulation, and the results includes numerical applications.

2. The displacement wave vector in a randomly heterogeneous medium

We have the relationship needed to derive an expression for the average displacement in the frequency domain. The average coefficient of the displacement associated with the combination of the random materials properties is

$$\langle u(x,\omega)\rangle = u_0(x,\omega) + \langle u_{RR}(x,\omega)\rangle\sigma_R^2 + \langle u_{RA}(x,\omega)\rangle\sigma_R\sigma_A + \langle u_{AA}(x,\omega)\rangle\sigma_A^2 + \mathrm{O}(\sigma^3). \quad (1)$$

This result can be obtained by the solution of the wave equations that are formed by Eqs. (A-10), (A-11) and (A-12). In this case we use the source functions given by Eqs. (A-17), (A-18) and (A-19). Thus, these new equations are given by

$$\left[\omega^2 + V_0^2\frac{\partial^2}{\partial x^2}\right]\langle u_{RR}(x,\omega)\rangle = \langle f_{RR}(x,\omega)\rangle = -\frac{1}{2}\omega^2(C(\omega)+1)u_0(x,\omega), \quad (2a)$$

$$\left[\omega^2 + V_0^2\frac{\partial^2}{\partial x^2}\right]\langle u_{AA}(x,\omega)\rangle = \langle f_{AA}(x,\omega)\rangle = -\frac{1}{2}\omega^2(C(\omega)+1)u_0(x,\omega), \quad (2b)$$

$$\left[\omega^2 + V_0^2\frac{\partial^2}{\partial x^2}\right]\langle u_{RA}(x,\omega)\rangle = \langle f_{RA}(x,\omega)\rangle = -\omega^2\tilde{r}C(\omega)u_0(x,\omega). \quad (2c)$$

The most appropriate method for solving Eqs. (2a), (2b) and (2c) is to decompose each term as a product of plane wave u_0 times an amplitude. That is,

$$\langle u_{RR}(x,\omega)\rangle = U_{RR}(x,\omega)u_0(x,\omega), \quad (3a)$$

$$\langle u_{AA}(x,\omega)\rangle = U_{AA}(x,\omega)u_0(x,\omega), \quad (3b)$$

$$\langle u_{RA}(x,\omega)\rangle = U_{RA}(x,\omega)u_0(x,\omega). \quad (3c)$$

Thus, it follows from Eqs. (2a), (2b) and (2c) that

$$V_0^2\left[\frac{\partial^2 U_{RR}}{\partial x^2} - 2j\frac{\omega}{V_0}\frac{\partial U_{RR}}{\partial x}\right] = -\frac{1}{2}\omega^2(C(\omega)+1), \quad (4a)$$

$$V_0^2\left[\frac{\partial^2 U_{AA}}{\partial x^2} - 2j\frac{\omega}{V_0}\frac{\partial U_{AA}}{\partial x}\right] = -\frac{1}{2}\omega^2(C(\omega)+1), \quad (4b)$$

$$V_0^2\left[\frac{\partial^2 U_{RA}}{\partial x^2} - 2j\frac{\omega}{V_0}\frac{\partial U_{RA}}{\partial x}\right] = -\omega^2\tilde{r}C(\omega). \quad (4c)$$

Next, define

$$F_1(\omega) = -\frac{1}{2}\omega^2(C(\omega)+1), \quad (5)$$

$$F_2(\omega) = -\omega^2\tilde{r}C(\omega). \quad (6)$$

Then, if we further define

$$E_{RR}(x,\omega) = \frac{\partial U_{RR}}{\partial x}, \quad (7a)$$

$$E_{AA}(x,\omega) = \frac{\partial U_{AA}}{\partial x}, \quad (7b)$$

$$E_{RA}(x,\omega) = \frac{\partial U_{RA}}{\partial x}. \quad (7c)$$

It follows that Eqs. (4a), (4b) and (4c) may be written

$$V_0^2\left(\frac{\partial E_{RR}}{\partial x} - 2j\frac{\omega}{V_0}E_{RR}\right) = F_1(\omega), \quad (8a)$$

$$V_0^2\left(\frac{\partial E_{AA}}{\partial x} - 2j\frac{\omega}{V_0}E_{AA}\right) = F_1(\omega), \quad (8b)$$

$$V_0^2\left(\frac{\partial E_{RA}}{\partial x} - 2j\frac{\omega}{V_0}E_{RA}\right) = F_2(\omega). \quad (8c)$$

The solutions to these equations are found to be

$$U_{RR}(x,\omega) = U_0^{RR}(\omega) + U_1^{RR}(\omega)e^{2jk_0x} + \frac{jF_1(\omega)}{2\omega V_0}x, \tag{9a}$$

$$U_{AA}(x,\omega) = U_0^{AA}(\omega) + U_1^{AA}(\omega)e^{2jk_0x} + \frac{jF_1(\omega)}{2\omega V_0}x, \tag{9b}$$

$$U_{RA}(x,\omega) = U_0^{RA}(\omega) + U_1^{RR}(\omega)e^{2jk_0x} + \frac{jF_2(\omega)}{2\omega V_0}x. \tag{9c}$$

The solutions given by Eqs. (9a), (9b) and (9c) will converge by applying the condition,

$$U_1^{RR}(\omega) = U_1^{AA}(\omega) = U_1^{RA}(\omega) = 0. \tag{10}$$

Since $U_0^{RR}(\omega)$, $U_0^{AA}(\omega)$, and $U_0^{RA}(\omega)$ are arbitrary functions of frequency we choose for convenience to be,

$$U_0^{RR}(\omega) = U_0^{AA}(\omega) = \frac{F_1(\omega)}{2\omega^2}, \tag{11a,b}$$

and

$$U_0^{RA}(\omega) = \frac{F_2(\omega)}{2\omega^2}. \tag{11c}$$

This then gives the solution

$$U_{RR}(x,\omega) = \frac{F_1(\omega)}{2\omega^2}\left[1 + j\frac{\omega}{V_0}x\right] = U_{AA}(x,\omega), \tag{12a,b}$$

$$U_{RA}(x,\omega) = \frac{F_2(\omega)}{2\omega^2}\left[1 + j\frac{\omega}{V_0}x\right], \tag{12c}$$

which is the same as

$$U_{RR}(x,\omega) = -\frac{1}{4}(C(\omega)+1)[1 + jk_0x] = U_{AA}(x,\omega), \tag{13a,b}$$

$$U_{RA}(x,\omega) = -\frac{1}{2}\tilde{r}C(\omega)[1 + jk_0x]. \tag{13c}$$

Substituting into Eqs. (3a), (3b), (3c) and (1)) gives

$$\begin{aligned}\langle u(x,\omega)\rangle &= u_0(x,\omega) - \frac{1}{4}(C(\omega)+1) \times [1 + jk_0x](\sigma_R^2 + \sigma_A^2)u_0(x,\omega) - \frac{1}{2}\tilde{r}C(\omega)[1 + jk_0x] \times (\sigma_R\sigma_A)u_0(x,\omega)\\ &= u_0(x,\omega)\left\{1 - \left[\frac{1}{4}(C(\omega)+1) \times (\sigma_R^2 + \sigma_A^2) + \frac{1}{2}\tilde{r}C(\omega)\sigma_R\sigma_A\right]\right\}\\ &\quad - jk_0x\left\{1 - \left[\frac{1}{4}(C(\omega)+1) \times (\sigma_R^2 + \sigma_A^2) + \frac{1}{2}\tilde{r}C(\omega)\sigma_R\sigma_A\right]\right\}.\end{aligned} \tag{14}$$

The solution expressed by Eq. (14) is unphysical in its stated form in the sense that it contains ''secular'' terms proportional to x. However, recognizing that the terms $O(\sigma_R^2)$, $O(\sigma_A^2)$ and $O(\sigma_R\sigma_A)$ were all obtained by perturbation (Taylor expansion), we may regard Eq. (14) as a Taylor expansion. The more natural expansion would be one of exponential character. However, similar reasoning was used by Landau et al. (1984) to improve the stochastic estimate of the effective dielectric constant of a mixture. This procedure may be called the ‘‘Landau extrapolation’’, or more specifically in the present case, the ‘‘exponential extrapolation’’. Thus, using the exponential extrapolation technique, we may write

$$\langle u(x,\omega)\rangle = u_0 e^{-jk_0 x}\{e^{-d(\omega)(1+jk_0 x)}\}, \tag{15}$$

where

$$d(\omega) = \frac{1}{4}(C(\omega)+1)(\sigma_R^2+\sigma_A^2) + \frac{1}{2}\tilde{r}C(\omega)\sigma_R\sigma_A. \tag{16}$$

We recognize that the integral $C(\omega)$ given by Eq. (A-16) has real and imaginary parts

$$C_R(\omega) = \mathrm{Re}\{C(\omega)\} = P\int_{-\infty}^{\infty}\frac{k_0(\omega)}{k-2k_0(\omega)}S(k)\mathrm{d}k, \tag{17a}$$

$$C_I(\omega) = \mathrm{Im}\{C(\omega)\} = -\pi k_0(\omega)S(2k_0(\omega)). \tag{17b}$$

Therefore, it follows that

$$d_R(\omega) = \frac{1}{4}(C_R(\omega)+1)(\sigma_R^2+\sigma_A^2) + \frac{1}{2}\tilde{r}C_R(\omega)\sigma_R\sigma_A, \tag{18a}$$

$$d_I(\omega) = \frac{1}{4}(C_I(\omega))(\sigma_R^2+\sigma_A^2) + \frac{1}{2}\tilde{r}C_I(\omega)\sigma_R\sigma_A \tag{18b}$$

and Eq. (15) may be written as

$$\langle u_0(x,\omega)\rangle = \{u_0 e^{-d(\omega)}\}e^{-jk_0 x(1+d(\omega))} \tag{19a}$$

or

$$\langle u_0(x,\omega)\rangle = \{u_0(\omega)e^{-d(\omega)}\}e^{-jk_0 x(1+d_R(\omega)+jd_I(\omega))} \tag{19b}$$

or

$$\langle u_0(x,\omega)\rangle = \{u_0(\omega)e^{-d(\omega)}\}e^{+d_I(\omega)k_0 x}e^{-jk_0 x(1+d_R(\omega))}. \tag{20}$$

In Eq. (20), we note that $d_I(\omega)$ is negative because $C_I(\omega) < 0$. It then follows that the attenuation coefficient of the displacement wave is

$$\eta_d = k_0|d_I(\omega)| = \frac{k_0^2\pi}{4}S(2k_0(\omega)) \times\left[(\sigma_R^2+\sigma_A^2)+2\tilde{r}\sigma_R\sigma_A\right]. \tag{21}$$

This form for the attenuation coefficient is quite similar to that derived by Shapiro and Zien (1993) and Shapiro et al. (1994), who also report attenuation proportional to $k^2S(2k)$. The exact form of the other results must vary because of the different notation and assumptions in the two approaches.

The real part of the effective wave vector for the displacement wave is

$$\mathrm{Re}[K_d] = k_0(1+d_R(\omega)), \tag{22}$$

which, after substituting Eqs. (18a) and (18b) becomes

$$\mathrm{Re}[K_d] = k_0\left\{1+\frac{1}{4}(1+C_R(\omega))(\sigma_R^2+\sigma_A^2) + \frac{1}{2}\tilde{r}C_R(\omega)\sigma_R\sigma_A\right\}. \tag{23}$$

Thus, the effective wave number is shown to change in a frequency-dependent manner by an amount proportional to σ^2. The effective velocity for a given frequency must correspondingly change by a proportional amount. It is known (Aki and Richards, 1980) that the Hilbert transform of the attenuation must yield the frequency times the slowness dispersion. Inspection of Eqs. (17a) and (17b) shows that C_R is the Hilbert transform of C_I and, thus, the dispersion behavior inherent in Eq. (23) is the causal complement to the attenuation behavior of Eq. (21).

The effective wave propagation vector derived in this section is used below to develop practical expressions for the attenuation and phase velocity for predicting scattering and intrinsic effects.

3. Introduction of intrinsic attenuation

The key results of this development are outlined here. The wave field for a plane wave passing through the stochastic region is given by

$$\langle u(x,\omega)\rangle = u_0(\omega)e^{-jk_0x}e^{-jd(\omega)k_0x} \tag{24}$$

where k_0 is the unperturbed wave number, ω/V_0, and d has real and imaginary parts which are given by

$$d_R(\omega) = \frac{1}{4}(1 + C_R(\omega))(\sigma_R^2 + \sigma_A^2) + \frac{1}{2}C_R(\omega)\sigma_R\sigma_A,$$

and

$$d_I(\omega) = \frac{1}{4}C_I(\omega)(\sigma_R^2 + \sigma_A^2) + \frac{1}{2}\tilde{r}C_I(\omega)\sigma_R\sigma_A. \tag{25}$$

In Eq. (25), σ_R and σ_A are the nondimensionalized standard deviations of the density and stiffness distributions, and $C_R(\omega)$ and $C_I(\omega)$ are the real and imaginary parts of the integral

$$C(\omega) = \int_{-\infty}^{\infty}\frac{k_0}{k-2k_0}S(k)dk. \tag{26}$$

The function $S(k)$ is the spectral density of the perturbations in the density and stiffness.

3.1. Addition of intrinsic attenuation

One of the main conceptual difficulties in the theory development is the solution of the $C(\omega)$ integral, Eq. (26) above. While the principal value formulation seems to provide useful results, having a complex integral $C(\omega)$ creates difficulties in the understanding of the basic concept of attenuation. In this section, we show that the inclusion of intrinsic (anelastic, or non-scattering based) attenuation to the problem resolves this difficulty.

Intrinsic attenuation may be added to the wave equation by considering that k_0 is complex. That is,

$$k_0 = k_R - jk_I, \tag{27}$$

where $k_R = \omega/V_0$ and $k_I = \omega/(2V_0Q)$ are real numbers. The intrinsic attenuation Q is allowed to vary with frequency, but must be positive. Given this definition, Eq. (26) may be rewritten as

$$C(\omega) = \int_{-\infty}^{\infty}\frac{k_R - jk_I}{k-2(k_R-jk_I)}S(k)dk. \tag{28}$$

This form is separable into real and imaginary parts of $C(\omega)$:

$$C_R(\omega) = \int_{-\infty}^{\infty}\frac{k_R(k-2k_R)-2k_I^2}{(k-2k_R)^2+4k_I^2}S(k)dk,$$

and

$$C_{\mathrm{I}}(\omega) = -\int_{-\infty}^{\infty} \frac{k k_{\mathrm{I}}}{(k-2k_{\mathrm{R}})^2 + 4k_{\mathrm{I}}^2} S(k)\mathrm{d}k. \quad (29)$$

These two integrals have finite, real integrands and are readily integrable on a computer for reasonable choices of $S(k)$. An analytical solution for a particular choice of $S(k)$ is presented later. It is possible to show that for even, well-behaved functions $S(k)$,

$$C_{\mathrm{R}}(\omega \to 0) = C_{\mathrm{I}}(\omega \to 0) = 0,$$
$$C_{\mathrm{R}}(\omega \to \infty) = -1/2, \quad (30)$$
$$C_{\mathrm{I}}(\omega \to \infty) = 0.$$

Furthermore, it may be shown that for $\omega > 0$, $C_{\mathrm{I}}(\omega) < 0$.

Eqs. (24) and (25) represent the original displacement wave solution given in Parra et al. (1999). Substituting in the real and imaginary parts of k_0 yields a new equation in terms of k_{R} and k_{I}

$$\langle u(x,\omega)\rangle = u_0 \mathrm{e}^{-j(k_{\mathrm{R}} + d_{\mathrm{R}} k_{\mathrm{R}} + d_{\mathrm{I}} k_{\mathrm{I}})x - (k_{\mathrm{I}} + d_{\mathrm{R}} k_{\mathrm{I}} - d_{\mathrm{I}} k_{\mathrm{R}})x}. \quad (31)$$

Recall that k_{R} and k_{I} are both defined to be positive numbers, and d_{I} is negative because C_{I} is negative. From this equation, it is apparent that the effective wave number is given in the first exponential, while the effective attenuation is given in the second exponential. The total dispersion then has the form

$$V_{\mathrm{eff}}(\omega) = \frac{V_0(\omega)}{1 + d_{\mathrm{R}}(\omega) + d_{\mathrm{I}}(\omega)/2Q(\omega)}, \quad (32)$$

where $V_0(\omega)$ includes the effects of intrinsic dispersion due to $Q(\omega)$. At the same time, the total attenuation is given by

$$Q_{\mathrm{eff}}(\omega) = \frac{1}{2}\,\frac{2Q(\omega)(1 + d_{\mathrm{R}}(\omega)) + d_{\mathrm{I}}(\omega)}{(1 + d_{\mathrm{R}}(\omega)) - 2Q(\omega)d_{\mathrm{I}}(\omega)}, \quad (33)$$

where again $Q(\omega)$ represents the effects of intrinsic dispersion only. In the limiting case that intrinsic dispersion is negligible (i.e., $Q \to \infty$, the effective properties reduce to

$$V_{\mathrm{eff}}(\omega) = V_0(1 - d_{\mathrm{R}}(\omega)),$$
$$Q_{\mathrm{eff}}(\omega) = \frac{1 + d_{\mathrm{R}}(\omega)}{2|d_{\mathrm{I}}(\omega)|}, \quad (34)$$

in the limit of second-order accuracy, and recalling that $d_{\mathrm{I}} < 0$.

3.2. Influence of second-order average properties and limiting behavior

The exponential form for the density and stiffness fluctuations is a convenient, compact, and positive definite form for the medium properties. It is worth noting, however, that the mean density and stiffness are not ρ_0 and α_0, but are affected by the exponential skew in the fluctuation distribution. Since the density and stiffness have the same functional form, only the density will be examined. The actual mean density is given by

$$\langle \rho \rangle = \rho_0 \int_{-\infty}^{\infty} P(R')\mathrm{e}^{\sigma_R R'}\mathrm{d}R', \quad (35)$$

where $P(R')$ is the probability density of a zero-mean, unit variance, Gaussian variable. Expanding Eq. (35), we obtain

$$\langle \rho \rangle = \rho_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-R'^2/2}\mathrm{e}^{\sigma_R R'}\mathrm{d}R'. \quad (36)$$

To evaluate this integral we employ the method of completing the square, and make the substitution $z = R' - \sigma_R$. Now,

$$\langle \rho \rangle = \rho_0 \mathrm{e}^{\sigma_R^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\mathrm{d}z. \quad (37)$$

The integrand is now simply $P(z)$, and the integral of any probability density function over its entire range must be unity. Thus, the mean density, and by similar reasoning, the mean stiffness are

$$\langle \rho \rangle = \rho_0 \mathrm{e}^{\sigma_R^2/2},$$
$$\langle \alpha \rangle = \alpha_0 \mathrm{e}^{\sigma_A^2/2}. \quad (38)$$

Now it is known that in an inhomogeneous medium, the high frequency (or ray theoretical) slowness is equal to the mean slowness. That is,

$$V_{\mathrm{RT}}^{-1} = \langle V^{-1} \rangle = \left\langle \frac{\rho^{1/2}}{\alpha^{1/2}} \right\rangle. \tag{39}$$

This may also be evaluated analytically given the assumed exponential property distributions. Using the joint probability distribution function for a bivariate Gaussian distribution (Kalbfleisch, 1985) with covariance $\tilde{r}$ ($\tilde{r}^2 < 1$),

$$V_{\mathrm{RT}}^{-1} = \frac{\rho_0^{1/2}}{\alpha_0^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \times \frac{1}{\sqrt{1-\tilde{r}^2}} \exp\left(-\frac{1}{2(1-\tilde{r}^2)} \times \left[R'^2 + A'^2 + 2\tilde{r}R'A' \right] \right) \times \exp\left(\frac{1}{2} (\sigma_R R' - \sigma_A A') \right) \mathrm{d}R' \mathrm{d}A'. \tag{40}$$

Holding A' constant in Eq. (40) for the moment, we make the substitution

$$z = \frac{R'}{\sqrt{1-\tilde{r}^2}} + \sqrt{1-\tilde{r}^2} \left(\frac{\tilde{r}A'}{1-\tilde{r}^2} - \frac{1}{2}\sigma_R \right), \tag{41}$$

to yield

$$V_{\mathrm{RT}}^{-1} = V_0^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \times \exp\left(-\frac{A'^2}{2(1-\tilde{r}^2)} - \frac{\sigma_A A'}{2} \right) \times \exp\left(-\frac{z^2}{2} \right) \exp\left(\frac{1}{2}(1-\tilde{r}^2) \times \left(\frac{\tilde{r}A'}{1-\tilde{r}^2} - \frac{\sigma_R}{2} \right)^2 \right) \mathrm{d}z \mathrm{d}A'. \tag{42}$$

The integral in z is now straightforward. The integral in A' may be evaluated by making the substitution

$$y = A' + \frac{1}{2}(\sigma_A + \tilde{r}\sigma_R). \tag{43}$$

After some algebra, the remaining expression simplifies to

$$V_{\mathrm{RT}}^{-1} = V_0^{-1} \exp\left(\frac{1}{8}\sigma_R^2 + \frac{1}{8}\sigma_A^2 + \frac{1}{4}\sigma_R\sigma_A \right) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \times \exp\left(-\frac{1}{2}z^2 - \frac{1}{2}y^2 \right) \mathrm{d}y \mathrm{d}z. \tag{44}$$

The double integral evaluates to unity. By taking a series expansion (to second order) of the remaining exponential, we find

$$V_{\mathrm{RT}}^{-1} = V_0^{-1} \left(1 + \frac{1}{8}\sigma_R^2 + \frac{1}{8}\sigma_A^2 + \frac{1}{4}\tilde{r}\sigma_R\sigma_A \right). \tag{45}$$

This is in an exact correspondence with the perturbation theory development, given in the high frequency limit that $C_{\mathrm{R}}(\omega) = -1/2$ and $C_{\mathrm{I}}(\omega) = 0$.

The low frequency (effective medium theory) limiting velocity for a statistically stationary medium was determined by Backus (1962). For the one-dimensional case considered here, this is given simply by

$$V_{\mathrm{EMT}}^{-2} = \langle \rho \rangle \left\langle \frac{1}{\alpha} \right\rangle. \tag{46}$$

The expression for the mean value of ρ is given above and the mean value of $1/\alpha$ may be calculated similarly. Then,

$$V_{\mathrm{EMT}}^{-1} = V_0^{-1} \mathrm{e}^{\sigma_R^2/4} \mathrm{e}^{\sigma_A^2/4}, \tag{47}$$

which is equivalent with second-order accuracy to

$$V_{\mathrm{EMT}}^{-1} = V_0^{-1}\left(1 + \frac{\sigma_R^2}{4} + \frac{\sigma_A^2}{4}\right). \tag{48}$$

This is itself equivalent to the perturbation theory prediction in the low frequency limit of $C_{\mathrm{R}}(\omega) = C_{\mathrm{I}}(\omega) = 0$. Thus, we have shown that in the limiting cases of high frequency and low frequency, the perturbation theory of wave propagation produces the correct dispersive velocities.

3.3. Influence of the spectral density and relations with other random medium theories

The intermediate frequency results of the perturbation theory depend on an integral over the spectral density, and so are intimately tied to the functional form of $S(k)$. Accordingly, it is worth examining the physical meaning of the spectral density, as well as geologically appropriate forms of $S(k)$. Furthermore, the spectral density (or a similar function) has been employed in previous work on wave propagation through random media, and comparisons between the perturbation theory results and results in the literature can be made.

First, the autocorrelation function for R' and A' will be defined as

$$\chi(a) = \langle R'(x) R'(x+a) \rangle = \langle A'(x) A'(x+a) \rangle. \tag{49}$$

Recall that R' and A' are defined as zero-mean random variates with identical spatial statistics. The requirement for statistical stationarity ensures that χ is not a function of x and only depends on the absolute value of a. It is not difficult to show that the spectral density, $S(k)$, is the Fourier transform of the autocorrelation function. This leads to two immediate restrictions on $S(k)$. First, since $\chi(a)$ is an even, real valued function, $S(k)$ must be an even, real valued function. Secondly, at $k=0$, S is given by

$$S(k=0) = \int_{-\infty}^{\infty} \chi(a) \mathrm{d}a, \tag{50}$$

which implies that $S(k=0)$ is positive and a maximum of $S(k)$ for most reasonable correlation functions χ.

The most common and most reasonable choice of correlation function is the exponential,

$$\chi(a) = \mathrm{e}^{-a/l}. \tag{51}$$

This form has both experimental justification (based on examination of velocity logs, Velzeboer, 1981; White et al., 1990), and theoretical justification (considering cyclical sedimentary transition as a Markov process, Velzeboer, 1981; Kerner, 1992). For this choice of correlation function, the spectral density has the simple form

$$S(k) = \frac{l}{\pi}\frac{1}{1+k^2 l^2}. \tag{52}$$

This is particularly convenient for two reasons. First, this form leads to a straightforward analytic solution of the $C(\omega)$ integral (Eq. (26)). Secondly, it leads to good correspondence with results from other techniques.

A brief solution to the $C(\omega)$ integral using the technique of contour integration is given here. The equation to solve is

$$C(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k_0}{z - 2k_0}\frac{1}{1+z^2 l^2} \mathrm{d}zl, \tag{53}$$

with k_0 complex and the integration to take place along the real axis. This integrand has three poles at $z = 2k_0$, and $z = \pm j/l$. It may be seen by inspection that the integrand goes to zero as z goes to ∞ regardless of the phase. Therefore, we may close the integration path in either the positive or negative j directions. The same result is obtained in either case, but since there is only one pole above the real axis (see Eq. (27)), we will close it in that direction. The result of the integral is then quite simply $2\pi j$ times the residue of the pole, or

$$C(\omega) = 2\pi j \frac{1}{\pi}\left(\frac{k_0}{\frac{j}{l} - 2k_0}\frac{1}{2j}\right). \tag{54}$$

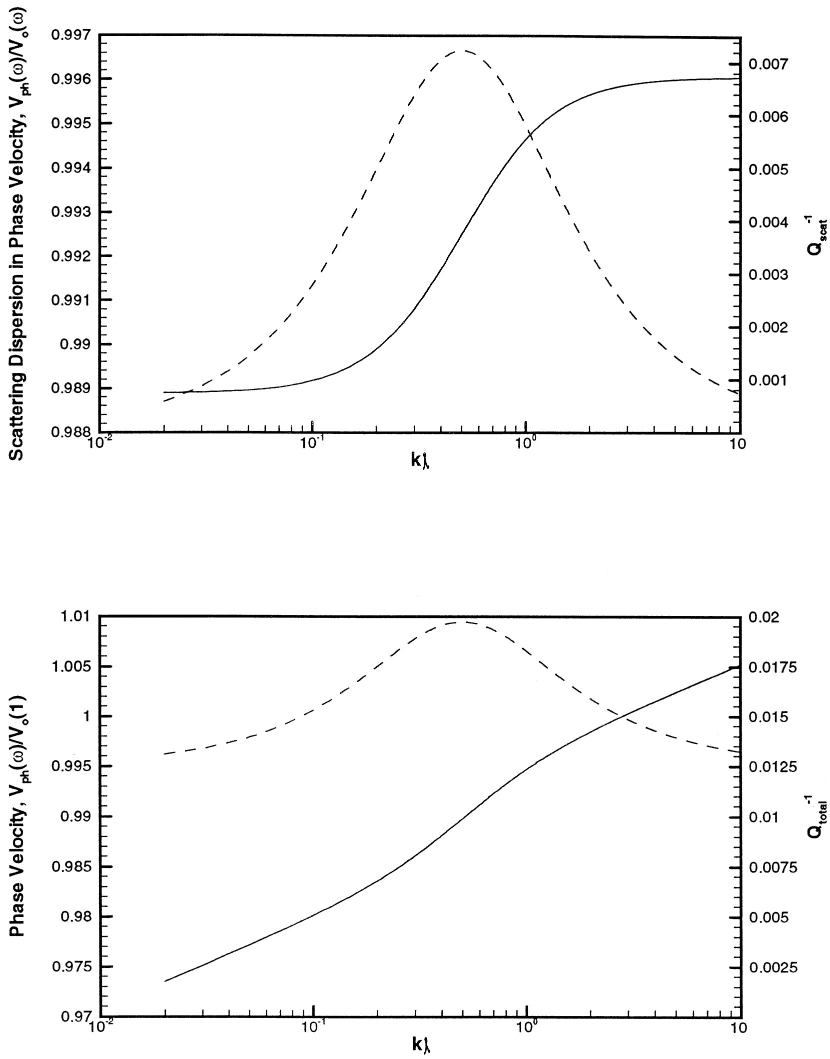

Fig. 1. Top: part of dispersion (solid line) and attenuation (dashed line) due only to scattering for a case with intrinsic attenuation corresponding to $Q(\omega)=80$. Bottom: total dispersion and attenuation due to both scattering and intrinsic effects. The scattering part of dispersion and attenuation appears superimposed on the uniform (linear) background.

Simplifying,

$$C(\omega) = -\frac{k_0 l(2k_0 l + j)}{4k_0^2 l^2 + 1}. \tag{55}$$

In the limit of no intrinsic attenuation ($k_\mathrm{I} \ll k_\mathrm{R}$), we can write this in terms of the spectral density,

$$C_\mathrm{R}(\omega) = -2\pi k_\mathrm{R}^2 l S(2k_\mathrm{R}),$$
$$C_\mathrm{I}(\omega) = -\pi k_\mathrm{R} S(2k_\mathrm{R}). \tag{56}$$

Expressions for velocity and attenuation in the limit of no intrinsic attenuation may be obtained by substituting Eqs. (52) and (56) into Eqs. (25) and (34). Now, since we have a solution valid in the limit of no intrinsic attenuation, it should be examined in comparison to the solutions of others. For this purpose note that the attenuation coefficient $\eta = k/2Q$ is proportional to kC_I. The exact constants of proportionality are dependent on the definitions made in the derivation, and so will not compare exactly with other work.

O'Doherty and Anstey (1971) state that the attenuation coefficient (is equal to the power spectrum of the autocorrelation function. In other words, $\eta \sim k^2 S(k)$. Shapiro et al. (1994) put this on a more mathematical footing, deriving an expression for η which is proportional to $k^2 S(2k)$. This result has the identical functional form to that derived here. Furthermore, Shapiro and Zien (1993) have combined this with the localization results of White et al. (1990) showing that η behaves as $\omega^2/(c_1\omega^2 + c_2)$. This implies that $S(k)$ has the form $1/(c_1 + c_2 k^2)$, which is the same as that desired.

4. Results

4.1. Numerical modeling

The following sample results will be based on a case with the following arbitrary parame-

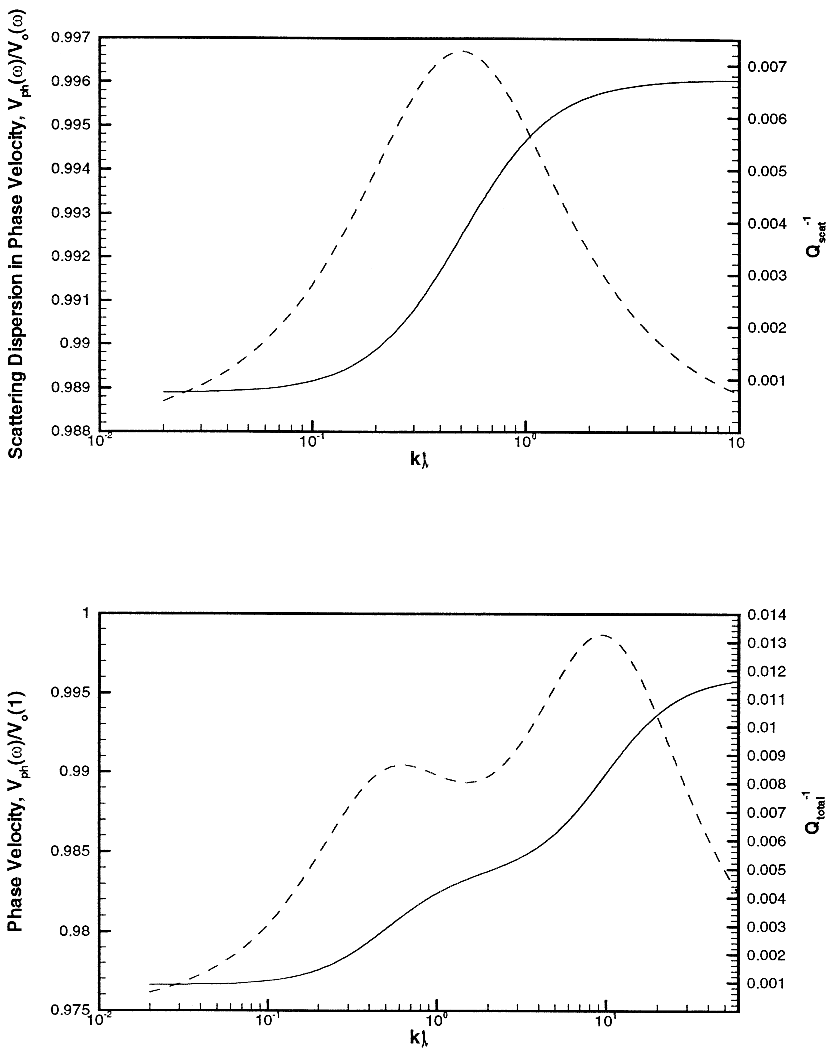

Fig. 2. Top: part of dispersion (solid line) and attenuation (dashed line) due only to scattering for a case with intrinsic attenuation only from a relation mechanism centered on $kl = 10$. Bottom: total dispersion and attenuation due to both scattering and intrinsic effects. Note that the scattering part and intrinsic part are easily distinguishable for this choice of parameters.

ters: $\sigma_R = 0.15$, $\sigma_A = 0.15$, $\tilde{r} = 0.3$, and characteristic length scale l. Changes in σ and $\tilde{r}$ generally only result in changes in magnitude rather than fundamental changes in the results. The exponential autocorrelation function is chosen, and wave numbers (or frequencies) are nondimensionalized by the correlation length to preserve generality. Since the correlation length only appears in the solution as a product of the wave number, the effect of changing the correlation length is only to move features to higher or lower frequencies. Fig. 1 shows the scattering part of dispersion and attenuation (as Q^{-1}) for the basic case with a constant intrinsic Q of 80. The lower part of the Fig. 1 shows the combined effects of both scattering and intrinsic dispersion and attenuation for this same case.

More interesting results come about when a frequency dependent intrinsic Q is introduced. This can be the result of a relaxation-type phenomenon as outlined in Aki and Richards (1980), or something similar such as the Biot or squirt flow mechanism (Parra, 1997). The mechanism introduced here will be of the general relaxation type. In this mechanism, the intrinsic Q is given by

$$\frac{1}{Q} = \frac{\omega(\tau_\varepsilon - \tau_\sigma)}{1 + \omega^2 \tau_\varepsilon \tau_\sigma}, \tag{57}$$

where τ_ε and τ_σ are characteristic relaxation time constants of the medium. Attenuation is at a maximum for $\omega = (\tau_\varepsilon \tau_\sigma)^{-1/2}$.

For a hypothetical medium with the base parameters above and a single attenuation peak of $Q = 80$ centered on $kl = 10$, the scattering based dispersion and attenuation is shown in Fig. 2. With the intrinsic part included, both components clearly show up independently in the combined dispersion and attenuation plot. As the correlation length varies, the center frequency of the scattering attenuation peak ($\omega = V/l$) will change with respect to the center frequency of the intrinsic attenuation peak. Thus, at times the scattering attenuation and dispersion may overlap the intrinsic attenuation and dispersion. At other times they may be sufficiently separable that an analyst can determine the type and magnitude of the underlying attenuation mechanism.

4.2. *Intrinsic and scattering dispersion of seismic waves in the Kankakee oil reservoir*

To illustrate the applicability of the present solution, we use crosswell seismic data recorded in the Kankakee Limestone Formation at the Buckhorn Test Site, IL. A heterogeneity that corresponds to the oil reservoir in the Kankakee Limestone Formation was detected using travel-time tomography by Saito (1991). The same heterogeneity was analyzed using well logs and dispersion and attenuation data by Parra (1995). Furthermore, the phase velocity data was modeled using the Biot and squirt flow mechanisms by Parra (1998) to predict the azimuthal permeability anisotropy. Although this model study predicted the fracture orientation in the Limestone Formation, it did not explain the presence of the heterogeneity in the experimental phase velocity curve that is associated with the reservoir.

In this paper we attempt to explain the effect of the heterogeneity using the present solution. This fractured zone may be better characterized by a two- or three-dimensional random model,

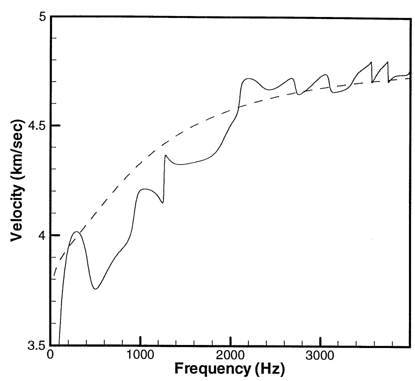

Fig. 3. Experimentally determined dispersion (solid line) and fitted random medium model dispersion (dashed line) for the Kankakee reservoir in Illinois.

but for the case of zero vertical offset compressional wave propagation, the one-dimensional formulation is probably adequate. In fact, we have obtained a reasonable fit to the experimental dispersion curve that was derived by Parra (1995) using the spectral ratio method. Fig. 3 shows the phase velocity data with theoretical model responses produced using a constant Q intrinsic model with $Q = 20$. The scattering response was obtained using a correlation length of about 0.5 m. It is important to note that while the model is a good fit qualitatively to the experimental data, many factors not present in the model do affect the field result. These include other sources of intrinsic dispersion such as squirt flow or spatially variable Q, and multidimensional effects. In addition, in the determination of phase velocity from the spectral ratio method there may be some unresolved effect from multiple arrivals, head waves, or mixed modes of propagation. The computed model Q (Fig. 4), however, matches the experimentally observed Q quite well with attenuation ranging from $Q = 5$ to $Q = 7$ over frequencies from 500 to 4000 Hz. This suggests that the strong dispersion observed for frequencies less than 2000 Hz is caused by the reservoir heterogeneity in the Kankakee Limestone Formation. On the other hand, the velocity distribution for frequencies greater than 2000 Hz is associated with the intrinsic properties if the reservoir (e.g., viscoelastic properties).

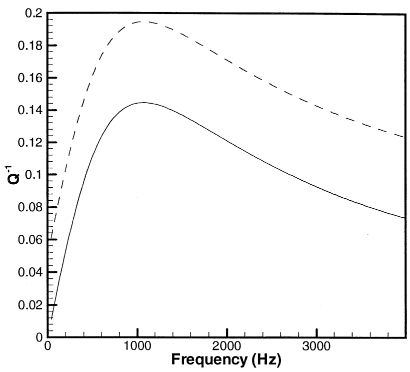

Fig. 4. Attenuation for the model of Fig. 3: attenuation due only to scattering (solid line) and combined scattering and intrinsic attenuation (dashed line). Intrinsic attenuation is modeled here by a frequency independent Q of 20. The resulting combined Q of 5–7 over the average of 500–4000 Hz is in good agreement with field measurements.

5. Conclusions

A second-order perturbation solution to the ensemble-averaged inhomogeneous wave equation was presented. Results of this solution have been shown to exhibit both dispersion and attenuation in a manner consistent with other work.

In particular, an analytical solution of the wave propagation vector was obtained. This relation was produced in terms of standard deviations of the density and Young's modulus as well as the cross-correlation coefficient and an integral that includes the spectral density and a kernel function.

The solution of the displacement and wave propagation vector can be extended to model two-dimensional heterogeneities by including the compressional and shear wave velocities of the medium. These extensions are presently under consideration and will be reported in the near future.

Our results show that inclusion of intrinsic attenuation in a random medium theory is straightforward, and that computation of the combined dispersion and attenuation can be readily performed. In addition, we give an analytical solution for a special case of the spectral density which is expected to correspond to a real sedimentary sequence. The relationship of a physical earth to the parameters underlying the theory is also discussed. Computed model results are shown, and comparisons are drawn to field data and other previously published theoretical work. Correspondence between the present work and previously established theoretical limiting behavior is within the expected second-order accuracy. Finally, the modeling of experimental phase velocity data show the scattering effects caused by the presence of the reservoir heterogeneity and the intrinsic effects caused the viscoelastic property of the reservoir.

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Appendix A. Second-order forcing functions of the heterogeneous wave equation

We write the displacement field $u(x,t)$ as an expansion in standard deviation parameters σ_R and σ_A, dropping terms higher than second-order in σ_R and σ_A. In Parra et al. (1999), Eq. 14 shows such an expansion, but where second-order terms are also dropped. Here, we retain the second-order terms. Thus, Taylor expansion of $u(\sigma_R,\sigma_A)$ about $(\sigma_R,\sigma_A)=(0,0)$ gives

$$u(\sigma_R,\sigma_A)=u_0+u_1+u_2+\mathrm{O}(\sigma^3), \qquad \text{(A-1)}$$

where the zero, first, and second-order parts are

$$u_0=u(0,0),$$
$$u_1=u_R\sigma_R+u_A\sigma_A, \qquad \text{(A-2)}$$
$$u_2=u_{RR}\sigma_R^2+u_{RA}\sigma_A\sigma_R+u_{AA}\sigma_A^2.$$

Parra et al. (1999) show that the first-order solution has zero-mean (in the sense of ensemble mean)

$$\langle u_1\rangle=0. \qquad \text{(A-3)}$$

However, this is not true in general for the second-order solution. Thus, a priori, we must write the mean displacement as

$$\begin{aligned}\langle u(x,t)\rangle &= u_0+\langle u_2\rangle+\mathrm{O}(\sigma^3)\\ &= u_0+\langle u_{RR}\rangle\sigma_R^2+\langle u_{RA}\rangle\sigma_R\sigma_A\\ &\quad+\langle u_{AA}\rangle\sigma_A^2+\mathrm{O}(\sigma^3).\end{aligned} \qquad \text{(A-4)}$$

To accomplish our objectives, we introduce the heterogeneous wave equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right), \qquad \text{(A-5)}$$

which, in terms of the log-coefficients $\alpha(x)$ and $\rho(x)$ (Parra et al., 1999), can be written as

$$\bar{\rho}_G e^{\sigma_R R'(x)}\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(\bar{\alpha}_G e^{\sigma_A A'(x)}\frac{\partial u}{\partial x}\right). \qquad \text{(A-6a)}$$

After expanding the exponential functions in Eq. (A-6a) using Taylor series expansion and including second-order terms such as

$$e^{\sigma_R R'(x)}=1+\sigma_R R'(x)+\frac{\sigma_R^2}{2}(R'(x))^2,$$

and

$$e^{\sigma_A A'(x)}=1+\sigma_A A'(x)+\frac{\sigma_A^2}{2}(A'(x))^2, \qquad \text{(A-6b)}$$

we insert Eq. (A-6b) into Eq. (A-6a) to obtain

$$\begin{aligned}&\left[1+\sigma_R R'(x)+\frac{\sigma_R^2}{2}(R'(x))^2\right]\frac{\partial^2 u}{\partial t^2}\\ &=V_0^2\left\{\frac{\partial^2 u}{\partial x^2}+\sigma_A\left(\frac{\partial A'}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right)\right.\\ &\quad\left.+\sigma_A^2\left[A'\frac{\partial A'}{\partial x}\frac{\partial u}{\partial x}+\frac{(A')^2}{2}\frac{\partial^2 u}{\partial x^2}\right]\right\}.\end{aligned} \qquad \text{(A-6c)}$$

In Eq. (A-6c), $R'(x)$ and $A'(x)$ are as zero-mean fluctuations with standard deviations σ_R and α_A, respectively (Yaglom, 1962). V_0 is the velocity of the plane wave incident and is given by $V_0=\sqrt{\bar{\alpha}_G/\bar{\rho}_G}$, in which $\bar{\rho}_G$ is the geometric mean density, and $\bar{\alpha}_G$ is the geometric mean of the stiffness coefficient, α, that is, $\alpha=\lambda+2\mu$.

Next, we obtain explicit expressions for the second-order displacement by extending the perturbation expansion to obtain equations for the second-order displacement terms, u_{RR}, u_{AA}, u_{RA} (respectively, σ_R^2, σ_A^2, $\sigma_R\alpha_A$). If we Fourier-transform with respect to time, and substitute Eq. (A-4) into Eq. (A-6c), we obtain the following second-order terms.

Order $(\sigma_R)^2$:

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)u_{RR}=f_{RR}$$

$$=-\omega^2\left(R'u_R+\frac{1}{2}R'^2u_0\right). \quad \text{(A-7)}$$

Order $(\sigma_A)^2$:

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)u_{AA}=f_{AA}$$

$$=-V_0^2\left\{\frac{\partial}{\partial x}\left(A'\frac{\partial u_A}{\partial x}+\frac{1}{2}\frac{\partial}{\partial x}\right)\right.$$

$$\left.\times\left((A')^2\frac{\partial u_0}{\partial x}\right)\right\}. \quad \text{(A-8)}$$

Order $\sigma_R\sigma_A$:

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)u_{RA}=f_{RA}$$

$$=-\left\{\omega^2R'u_A+V_0^2\frac{\partial}{\partial x}\left(A'\frac{\partial u_R}{\partial x}\right)\right\}. \quad \text{(A-9)}$$

Since we are only interested in the ensemble means of u_{RR}, u_{AA}, u_{RA}, we restrict ourselves to the mean equations obtained by applying the ensemble mean operator to both sides of Eqs. (A-7), (A-8) and (A-9). Taking into account that R' and A' were normalized to unit variance

$$\langle R'^2\rangle=\langle A'^2\rangle=1$$

we obtain, respectively,

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)\langle u_{RR}\rangle=\langle f_{RR}\rangle$$

$$=-\omega^2\left(\langle R'u_R\rangle+\frac{1}{2}u_0\right), \quad \text{(A-10)}$$

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)\langle u_{AA}\rangle=\langle f_{AA}\rangle$$

$$=-V_0^2\left\{\frac{\partial}{\partial x}\left\langle A'\frac{\partial u_A}{\partial x}\right\rangle+\frac{1}{2}\frac{\partial^2 u_0}{\partial x^2}\right\}, \quad \text{(A-11)}$$

$$\left(\omega^2+V_0^2\frac{\partial^2}{\partial x^2}\right)\langle u_{RA}\rangle=\langle f_{RA}\rangle$$

$$=-\left\{\omega^2\langle R'u_A\rangle+V_0^2\frac{\partial}{\partial x}\left\langle A'\frac{\partial u_R}{\partial x}\right\rangle\right\}. \quad \text{(A-12)}$$

We now recognize on the right-hand sides of Eqs. (A-10), (A-11) and (A-12) several terms that can be easily derived, namely

$$\langle R'u_R\rangle,\ \langle R'u_A\rangle,\ \left\langle A'\frac{\partial u_R}{\partial x}\right\rangle, \text{ and } \left\langle A'\frac{\partial u_A}{\partial x}\right\rangle.$$

Since the derivation of the statistical average of these terms are similar in nature, we present the development of only two of the terms. The first of these terms is

$$\langle R'u_R\rangle=\int\int e^{j(k'-k)x}\langle d\hat{U}_R(k')d\hat{R}^*(k)\rangle$$

where

$$\mathrm{d}\hat{U}_R(k') = U_0(\omega)\left(k_0^2/\left(k'^2 - k_0^2\right)\right)\mathrm{d}\hat{R}(k'+k_0).$$

Thus, using the stochastic solution in terms of random Fourier–Stieltjes increments shown in Parra et al. (1999), we can derive the statistical average of the product

$$\langle R' u_R\rangle = \int\int \mathrm{e}^{j(k'-k)x} U_0(\omega) \frac{k_0^2}{k'^2 - k_0^2} \times \langle \mathrm{d}\hat{R}(k'+k_0)\mathrm{d}\hat{R}^*(k)\rangle.$$

Next assuming, stationarily unrelated zero-mean random fields, we apply the relationship, $\langle \mathrm{d}\hat{R}(k'+k_0)\mathrm{d}\hat{R}^*(k)\rangle = S(k)\delta(k_0+k'-k)\mathrm{d}k\,\mathrm{d}k'$, to obtain,

$$\begin{aligned}
\langle R' u_R\rangle &= U_0(\omega)\int\int \mathrm{e}^{j(k'-k)x} \frac{k_0^2}{k'^2-k_0^2} S(k) \\
&\quad \times \delta(k_0+k'-k)\mathrm{d}k\mathrm{d}k' \\
&= U_0(\omega)\mathrm{e}^{-jk_0x} \\
&\quad \times \int_{-\infty}^{+\infty} \frac{k_0^2}{(k-k_0)^2 - k_0^2} S(k)\mathrm{d}k \\
&= \frac{1}{2}u_0(x,\omega) \\
&\quad \times \int_{-\infty}^{+\infty} \frac{k_0^2}{k^2 - 2k_0k} S(k)\mathrm{d}k \\
&= \frac{1}{2}u_0(x,\omega) \\
&\quad \times \int_{-\infty}^{+\infty} k_0\left(\frac{1}{k-2k_0} - \frac{1}{k}\right) S(k)\mathrm{d}k \\
&= \frac{1}{2}u_0(x,\omega)\int_{-\infty}^{+\infty} \frac{k_0}{k-2k_0} S(k)\mathrm{d}k.
\end{aligned} \tag{A-13}$$

In Eq. (A-13) the odd integrand term $S(k)/k$ is zero and has been eliminated. Similarly, the statistical average of the product $R'u_A$ is given by

$$\langle R' u_A\rangle = \int\int \mathrm{e}^{j(k'-k)x}\langle \mathrm{d}\hat{U}_A(k')\mathrm{d}\hat{R}^*(k)\rangle.$$

Its derivation is as follows

$$\begin{aligned}
\langle R' u_A\rangle &= U_0(\omega)\int\int \mathrm{e}^{j(k'-k)x} \frac{k_0^2}{k'^2-k_0^2}\frac{k'}{k_0} \\
&\quad \times \langle \mathrm{d}\hat{A}(k'+k_0)\mathrm{d}\hat{R}^*(k)\rangle \\
&= U_0(\omega)\int\int \mathrm{e}^{j(k'-k)x} \frac{k_0^2}{k'^2-k_0^2}\frac{k'}{k_0} \\
&\quad \times \tilde{r}S(k)\delta(k_0+k'-k)\mathrm{d}k\mathrm{d}k' \\
&= \tilde{r}U_0(\omega)\mathrm{e}^{-jk_0x}\int_{-\infty}^{+\infty} \frac{k_0^2}{(k-k_0)^2-k_0^2} \\
&\quad \times \frac{k-k_0}{k_0}S(k)\mathrm{d}k \\
&= \frac{1}{2}\tilde{r}u_0(x,\omega) \\
&\quad \times \int_{-\infty}^{+\infty}\left[\frac{k_0}{k-2k_0} + \frac{k_0}{k}\right]S(k)\mathrm{d}k,
\end{aligned}$$

where $\tilde{r}$ is the cross-correlation coefficient $(-1 \le \tilde{r} \le 1)$. Noting that the odd integrand term is zero, the statistical average of the above product is reduced to

$$\langle R' u_A\rangle = \frac{1}{2}\tilde{r}u_0(x,\omega)\int_{-\infty}^{+\infty}\left[\frac{k_0}{k-2k_0}\right]S(k)\mathrm{d}k. \tag{A-14}$$

Therefore, for a general random medium with spectrum $S(k)$ using the identities:

$$u_0(x,\omega) = U_0(\omega)\mathrm{e}^{-jk_0x};\ \text{for } (k_0 = \omega/V_0), \tag{A-15a}$$

$$\frac{\partial u_0}{\partial x} = -jk_0u_0, \tag{A-15b}$$

$$\frac{\partial^2 u_0}{\partial x^2} = -k_0^2u_0. \tag{A-15c}$$

Also, we define, for a general random medium,

$$C(\omega) = \int_{-\infty}^{+\infty}\frac{k_0(\omega)}{k-2k_0(\omega)}S(k)\mathrm{d}k, \tag{A-16}$$

where $k_0(\omega)$ is complex variable that includes the intrinsic attenuations, and $S(k)$ is the spectral density.

We can write the right-hand sides of Eqs. (A-10), (A-11) and (A-12) in terms of integral $C(\omega)$:

$$\langle f_{RR} \rangle = -\frac{1}{2}\omega^2 (C+1) u_0, \tag{A-17}$$

$$\langle f_{AA} \rangle = +\frac{1}{2} V_0^2 (C+1) \frac{\partial^2 u_0}{\partial x^2} = -\frac{1}{2}\omega^2 (C+1) u_0, \tag{A-18}$$

and

$$\langle f_{RA} \rangle = -\left\{ \omega^2 \tilde{r} u_0 C/2 + V_0^2 \frac{\partial}{\partial x}\left(-\tilde{r}\varepsilon_0 C/2\right) \right\} = -\left\{ \omega^2 \tilde{r} u_0 \frac{C}{2} + \omega^2 \tilde{r} u_0 \frac{C}{2} \right\} = -\omega^2 \tilde{r} C u_0. \tag{A-19}$$

Eqs. (A-17), (A-18) and (A-19) are relationships for the second-order forcing functions of the heterogeneous wave equation. These equations are used to determine explicit expressions for the heterogeneous wave displacement and the effective wave propagation vector.

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