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A stochastic wavefield solution of the acoustic wave equation based on the random Fourier–Stieltjes increments

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Abstract

In this paper we investigate the problem of compressional wave seismic propagation in random media. This problem is important because almost all geologic media is spatially heterogeneous by nature, consisting of a random agglomerate of many-sized rocks, soil and strata. In our formulation, a plane-harmonic seismic wave propagates in a medium having random material properties in the vertical direction. The random field representation is introduced through the intrinsic rock physical properties of the elastic medium. Each of these intrinsic properties is assumed to have a log-normal probability density function, and the random field representation is expressed in terms of these log-normal probability density functions. The constitutive law, the mass balance, and the moment balance equations are written in the Fourier–Stieltjes representation using random Lamé coefficients and random mass density. The stochastic wave equation is developed by introducing a perturbation approach based on an infinite series expansion of both random coefficients and the displacement solution in terms of σ-parameters (standard deviations of the random material properties). The method yields an integral representation of the displacement wavefield based on the Green's function. This representation is expressed in terms of the random rock physical properties of the medium. The key feature of this paper is that we have expressed the solution as a function of statistical parameters of 1D random medium, including the second order moments. Contrary to most previous derivations, the solutions can also simulate the coda and can be easily extended to simulate waves propagating in 2D and 3D random media. To test the displacement wave solution, synthetic seismograms and dispersion due to scattering effects were calculated for stiffness and density fluctuations of the random medium. This paper is the underlying foundation for the development of the effective propagation vector of acoustic waves in randomly heterogeneous media. This development is presented in a companion paper. In this case, an analytical expression is obtained using a second order perturbation solution. The solution is obtained in terms of the standard deviations of the density and the Young's modulus, respectively, as well as the cross-correlation coefficient and an integral that includes the spectral density and a kernel. In addition, this paper introduces practical expressions for the calculation of the effective attenuation and phase velocity of waves in randomly heterogeneous media. In this companion paper the solution is applied to interpret phase velocity curves that were obtained from interwell acoustic data recorded at Buckhorn test site, Illinois. The objective in this case is to be able to simulate the effect of scattering and intrinsic attenuation associated with acoustic waves in randomly heterogeneous media.

Keywords: Stochastic wavefield solution; Acoustic wave equation; Fourier–Stieltjes increments

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1. Introduction

The inherently complex nature of geological heterogeneities have produced an increasing interest in using statistical models for describing spatial heterogeneity, particularly in the treatment of subsurface contamination, oil and gas reservoir characterization, and seismic data processing. In the characterization of hydrocarbon reservoirs, seismic methods play an important role. A reservoir formation varies in many different scales: from scales much larger than seismic wavelengths down to scales that are much smaller. It has been shown that the small-scale variations can have a significant effect on the transmitted wavefield and can give rise to apparent attenuation and dispersion (O'Doherty and Anstey, 1971). In the context of subsurface hydrology and toxic waste contamination, extensive field investigations have shown that the hypothesis of random field heterogeneity is consistent with observed spatial distribution of material properties and with the resulting contamination patterns. A comprehensive review of field measurements, including both saturated and unsaturated, continuous and fractured media, can be found in Ababou (1991).

There is extensive background on wave propagation in heterogenous media, and on statistical studies of various wave propagation phenomena and transport processes, in the applied physics literature. Our survey of the literature indicates that heterogeneous wave equations have been treated by a variety of methods based either on approximate multiple-scale expansions for periodic media (Santosa and Symes, 1991) or on various perturbation methods for statistical continua (Hoffman, 1964; Keller, 1964). The statistical continuum approach has been used to characterize effective velocity and attenuation (Lerche and Petroy, 1986). Seismic wave propagation in random media in reference to dispersion and wave attenuation due to scatterers has been addressed by Korn (1993). Several authors have used statistical representations to describe small scale inhomogeneities in seismological studies (Kerner, 1992; Ikelle et al., 1993; Kneib and Kerner, 1993; Roth and Korn, 1993). In particular, these authors have used the finite-difference algorithm to simulate seismic wave propagation.

In this paper, we present a fundamental analysis of wave propagation in a randomly heterogeneous geologic medium by treating the one-dimensional stochastic problem in 1D space of plane-harmonic compressional wave propagation in a medium having random material properties distributed in the vertical direction. To demonstrate the applicability of the present solution, numerical models for a medium having random rock physical properties are formulated to calculate synthetic seismograms and related dispersion of scattering effects associated with such random medium characteristics.

The paper is divided into four parts: (Section 2) formulation of the stochastic wave equation; (Section 3) development of perturbation solutions for the 1D stochastic wave equation; (Section 4) numerical evaluation of the stochastic wavefields; and (Section 5) numerical results. The paper also includes two appendices: (Appendix A) Fourier space–Green's function solution using ''informal'' Fourier representations; and (Appendix B) expression of the stochastic solution in terms of random Fourier–Stieltjes increments.

2. Formulation of the stochastic wave equation

The constitutive law, the mass balance, and the momentum balance equations are introduced for the development of the stress and displacement solutions for random increments using the Fourier–Stieltjes representation. The random field representation is introduced through the intrinsic rock physical properties of the elastic medium. Each of these intrinsic properties is assumed to have a log-normal probability density function, and the random field representation is expressed in terms of the log-normal probability density functions.

2.1. Formulation of model problem / random coefficients

We assume that the constitutive law for stress (τ) and strain (ε) is given by the classical linear-isotropic relation with spatially variable Lamé coefficients $\lambda(x)$ and $\mu(x)$, both of which may be random variables. Thus,
constitutive law:

$$\tau = (\lambda(x) + 2\mu(x))\varepsilon, \tag{1}$$

or

$$\tau = \alpha(x)\varepsilon. \tag{2}$$

In addition, we assume that the density of the medium, $\rho(x)$, in the unconstrained (equilibrium) state, may also be randomly variable. Therefore, the linearized momentum equation is obtained from mass balance and momentum balance, as follows:
mass balance (density):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial\rho}{\partial t} + \frac{\partial(\rho V)}{\partial x} = 0, \tag{3}$$

$$V = \frac{\partial u}{\partial t} \quad \text{and} \quad \rho = \rho_0(x)\frac{1}{1+\varepsilon}, \tag{4}$$

where ρ is mass density, u is displacement, and V is phase velocity.

2.1.1. Momentum balance

Neglecting quadratic displacement effects (due to non-equilibrium variations of density governed by Eqs. (1)–(4), the (linearized) momentum is

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial\tau}{\partial x}, \tag{5}$$

where spatial variations of $\rho_0(x)$ correspond to material heterogeneity of density (at equilibrium). Inserting Eq. (2) in Eq. (5) and dropping the zero subscript in $\rho_0(x) \approx \rho(x)$, gives

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right). \tag{6}$$

Eq. (6) can be reduced to the solution of a plane wave propagating in 1D random geological medium having material heterogeneities randomly distributed in the x-direction (vertical). For this solution the following rock physical properties are considered

$$\rho(x) = \mathrm{e}^{R(x)}, \tag{7}$$

and

$$\alpha(x) = \mathrm{e}^{A(x)}, \tag{8}$$

where $R(x)$ and $A(x)$ are assumed to be Gaussian random fields (real-valued). We adopt here for convenience the singular set of random fields defined by Eqs. (7) and (8).

The mean values of the log-coefficients are the geometric means of the coefficients themselves. Thus,

$$R(x) = \ln\rho(x) = \bar{R} + \sigma_R R'(x), \tag{9a}$$

$$A(x) = \ln\alpha(x) = \bar{A} + \sigma_A A'(x), \tag{9b}$$

where $\langle R' \rangle = 0$, $\langle A' \rangle = 0$, and

$$\bar{R} = \ln(\bar{\rho}_G) \Leftrightarrow \bar{\rho}_G = \mathrm{e}^{\bar{R}}, \tag{10a}$$

$$\bar{A} = \ln(\bar{\alpha}_G) \Leftrightarrow \bar{\alpha}_G = \mathrm{e}^{\bar{A}}. \tag{10b}$$

2.2. Spectral representations of the random log-coefficients

The log-coefficients $R(x)$ and $A(x)$, or their zero-mean fluctuations $R'(x)$ and $A'(x)$ have spectral representations of the form (Yaglom, 1962; Vanmarcke, 1983)

$$R'(x) = \int \mathrm{e}^{jkx}\mathrm{d}\hat{R}(k), \tag{11a}$$

$$A'(x) = \int \mathrm{e}^{jkx}\mathrm{d}\hat{A}(k). \tag{11b}$$

We observe here that, for exceptionally smooth random fields, the somewhat unfamiliar Fourier–Stieltjes increments

$$\mathrm{d}\hat{R}(k), \quad \mathrm{d}\hat{A}(k) \tag{12a}$$

may be replaced by the more familiar forms

$$\tilde{R}(k)\mathrm{d}k, \quad \tilde{A}(k)\mathrm{d}k, \tag{12b}$$

where $\tilde{R}(k)$ and $\tilde{A}(k)$ are the (random) Fourier component of the random fields.

2.3. Remark about normalization

In Eqs. (9a) and (9b) we have expressed R' and A' in such a way that they are normalized Gaussian random fields with zero means and unit variances. (The actual variance of R and A are σ_R^2 and σ_A^2). Rewriting governing equations in terms of log-coefficients, the following are derived.

- Constitutive law

  Eq. (2):

$$\tau = \alpha(x)\varepsilon \rightarrow \tau = \bar{\alpha}_{\mathrm{G}} \mathrm{e}^{\sigma_A A'(x)}\varepsilon; \tag{13a}$$

- Mass balance (density)

  Eqs. (3) and (4): First, simplify by assuming $\rho(x) \approx \rho_0(x)$. We then express the variable density as

$$\rho(x) \approx \rho_0(x) = \bar{\rho}_{\mathrm{G}} \mathrm{e}^{\sigma_R R'(x)}; \tag{13b}$$

- Momentum balance:

  Eq. (5):

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right),$$

$$\Rightarrow \mathrm{e}^{\sigma_R R'(x)}\frac{\partial^2 u}{\partial t^2} = V_0^2 \mathrm{e}^{\sigma_A A'(x)} \times \left(\sigma_A \frac{\partial A'}{\partial x} + \frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x} = 0. \tag{13c}$$

This result assumes that $V_0 = (\bar{\alpha}_{\mathrm{G}}/\bar{\rho}_{\mathrm{G}})^{1/2}$.

Eq. (13c) is the 1D stochastic wave equation containing the zero-mean fluctuations $R'(x)$ and $A'(x)$, and their standard deviations σ_R and σ_A, respectively. To determine the random wavefield, Eq. (13c) is solved using a perturbation approach.

3. Development of perturbation solutions for the 1D stochastic wave equation

The stochastic wave equation is solved by introducing a perturbation approach based on an infinite series expansion of both the random coefficients and the displacement solution in terms of σ-parameters (standard deviations of random material properties). This perturbation approach is based on the work of Ababou and Gehlar (1990). The method yields the first-order approximation of the 1D particle displacement solution for an incident plane compressional wave in an unbounded heterogeneous medium. The stochastic displacement wavefield is expressed in closed form in terms of the incident deterministic plane wave and in terms of the random material properties of the medium.

3.1. Methodology

The perturbation equations are obtained by inserting the normalized forms of the random coefficients (Eqs. (9a) and (9b)) into the governing momentum equation (Eq. (5)), and expanding the result in terms of the σ-parameters

σ_R = standard deviation of ln $\rho(x)$,

σ_A = standard deviation of ln $\alpha(x)$.

Note that, since there is more than one parameter, a multi-parameter expansion is required. In the following sections, we will proceed as follows:

1. expand the dependent variable (displacement u) to first-order in terms of the σ-parameters, thus neglecting terms of order $O(\sigma^2)$ or higher;
2. solve the resulting stochastic wave equation (a classical wave equation except for the presence of a spatially distributed random source term);
3. perform a variety of statistical analyses on the stochastic displacement wave-field and insert the results into the (random) constitu-

tive equations to obtain effective stress–strain relations in terms of the mean stress and strain wavefields.

3.2. Development of perturbation equations

Since the solution $u(x,t)$ of Eq. (5) must depend on the material property parameters σ_R and σ_A, we choose a Taylor-expansion of $u = u(x,t;\ \sigma_R,\sigma_A)$ around the parameter values $\sigma_R = 0$ and $\sigma_A = 0$. Dropping the (x,t) dependence from our notations (for convenience), we have

$$u(\sigma_R,\sigma_A) = u_0 + u_R\sigma_R + u_A\sigma_A + O(\sigma^2), \tag{14}$$

where $O(\sigma^2)$ stands for $O(\sigma_R^2) + O(\sigma_A^2) + O(\sigma_A\sigma_R)$, and, by construction, we note the ‘‘superposition relations’’:

$$u(0,0) = u_0, \tag{15a}$$

$$u(\sigma_R,0) = u_0 + u_R\sigma_R + O(\sigma^2), \tag{15b}$$

$$u(0,\sigma_A) = u_0 + u_A\sigma_A + O(\sigma^2), \tag{15c}$$

and

$$u(\sigma_R,\sigma_A) = u(0,0) + \{u(\sigma_R,0) - u(0,0)\} + \{u(0,\sigma_A) - u(0,0)\} + O(\sigma^2). \tag{16}$$

Eq. (16) specifically indicates that the total displacement wave can be obtained by a single superposition of $u(0,0)$, $u(\sigma_R,0)$, and $u(0,\sigma_A)$.

Note the (obvious) meaning of these ‘‘elementary wave-fields’’:

- $u(0,0) =$ deterministic wave-field for constant coefficient values $R = \bar{R}$ and $A = \bar{A}$, or equivalently, $\rho = \rho_G$ and $\alpha = \bar{\alpha}_G$ (geometric mean values).
- $u(\sigma_R,0) =$ stochastic wave field for random $R(x)$ (i.e., $\rho(x)$) but constant $A(x)$ (i.e., $\alpha(x) = \bar{\alpha}_G$).
- $u(0,\sigma_A) =$ stochastic wave field for constant $R(x)$ (i.e., $\rho(x) = \bar{\rho}_G$), but random $A(x)$ (i.e., $\alpha(x)$).

We now proceed to develop separate equations for each of the sub-problems just defined.

3.2.1. First sub-problem

Deterministic $u(0,0) \equiv u_0$, corresponding to $\sigma_R = \sigma_A = 0$ and $u(0,0) = u_0$ is solution of the equation:

$$\mathcal{L}_0(u_0) = 0, \tag{17}$$

with d'Alembertian operator

$$\mathcal{L}_0 = \frac{\partial^2}{\partial t^2} - V_0^2\frac{\partial^2}{\partial x^2}, \text{ in which } V_0 = \sqrt{\frac{\bar{\alpha}_G}{\bar{\rho}_G}}. \tag{18}$$

3.2.2. Second sub-problem

Random $u(\sigma_R,0) \equiv u_0 + \sigma_R u_R$, with $\sigma_A = 0$: (19)

Inserting Eq. (19) in the momentum equation (see Eqs. (5) and (13c)) yields

$$e^{\sigma_R R'(x)}\frac{\partial^2(u_0 + \sigma_R u_R)}{\partial t^2} - (V_0^2)\frac{\partial^2(u_0 + \sigma_R u_R)}{\partial x^2} = 0. \tag{20a}$$

Expanding the exponential in Eq. (20a) (the Taylor series of e^y is convergent for all finite values of y),

$$e^{\sigma_R R'(x)} \cong 1 + \sigma_R R'(x) + O(\sigma_R^2), \tag{20b}$$

and inserting the expansion of Eq. (20b) in Eq. (20a) gives, after rearranging,

$$\left(\frac{\partial^2\cdot}{\partial t^2} - V_0^2\frac{\partial^2\cdot}{\partial x^2}\right)u_0 + \sigma_R\left\{\left(\frac{\partial^2\cdot}{\partial t^2} - V_0^2\frac{\partial^2\cdot}{\partial x^2}\right)u_R + R'(x)\frac{\partial^2 u_0}{\partial t^2}\right\} + O(\sigma_R^2) = 0. \tag{21}$$

Recognizing that the first term vanishes ($\mathcal{L}(u_0) \equiv 0$) and dropping all terms of order higher than $O(\sigma_R)$ gives the first-order approximate equation

$$\left(\frac{\partial^2}{\partial t^2} - V_0^2\frac{\partial^2}{\partial x^2}\right)u_R = -R'(x)\frac{\partial^2 u_0}{\partial t^2} = -f_R. \tag{22}$$

This equation is a stochastic wave equation for the displacement $u_R(x,t)$. The stochasticity arises only from a spatially distributed random source term $f_R(x,t)$, given by

$$f_R(x,t) = R'(x)\frac{\partial^2 u_0}{\partial t^2}. \tag{23}$$

3.2.3. Third sub-problem

Random $u(0,\sigma_A) = u_0 + \sigma_A u_A$, with $\sigma_R = 0$. For this problem, we obtain by the same method used above

$$\begin{aligned}0 &= \frac{\partial^2}{\partial t^2}(u_0+\sigma_A u_A) - V_0^2 e^{\sigma_A A'(x)} \\ &\quad\times\left(\sigma_A\frac{\partial A'}{\partial x}+\frac{\partial}{\partial x}\right)\frac{\partial(u_0+\sigma_A u_A)}{\partial x} \\ &= \left[\frac{\partial^2}{\partial t^2}u_0 - V_0^2\frac{\partial^2 u_0}{\partial x^2}\right] + \frac{\partial^2}{\partial t^2}\sigma_A u_A \\ &\quad - V_0^2\frac{\partial^2}{\partial x^2}\sigma_A u_A - V_0^2\sigma_A A'(x) \\ &\quad\times\frac{\partial^2}{\partial x^2}(u_0+\sigma_A u_A) - V_0^2(1+\sigma_A A'(x)) \\ &\quad\times\sigma_A\frac{\partial A'}{\partial x}\frac{\partial}{\partial x}(u_0+\sigma_A u_A),\end{aligned} \tag{24a}$$

after first replacing $e^{\sigma_A A'(x)} = 1 + \sigma_A A'(x) + O(\sigma_A^2)$. Thus, we obtain, after also using Eq. (5) and rearranging terms,

$$\begin{aligned}&\left(\frac{\partial^2}{\partial t^2} - V_0^2\frac{\partial^2}{\partial x^2}\right)u_A \\ &\quad = V_0^2\frac{\partial}{\partial x}\left(A'(x)\frac{\partial u_0}{\partial x}\right) = -f_A,\end{aligned} \tag{24b}$$

with the stochastic source term expressed by:

$$f_A(x,t) = -(V_0)^2\frac{\partial}{\partial x}\left(A'(x)\frac{\partial u_0}{\partial x}\right). \tag{25}$$

3.3. Fourier–Stieltjes representation

The next step in the analysis is to replace Eq. (B-6) (i.e., right-going plane wave), in the frequency-domain versions of Eqs. (23) and (25). This gives

$$\sigma_R:\ f_R(x,\omega) = -\omega^2 U_0(\omega)R'(x)e^{-jk_0x}; \tag{26a}$$

$$\sigma_A:\ f_A(x,\omega) = jk_0V_0^2U_0(\omega)\left\{\frac{\partial A'}{\partial x} - jk_0A'(x)\right\}e^{-jk_0x}. \tag{26b}$$

The expressions for the Fourier–Stieltjes (F–S) increments can be obtained in several different ways. The easiest procedure is to use the $f(x,\omega)$ formulas above (Eqs. (26a) and (26b)) and to apply the property of Eq. (B-3). For this, we define the following quantities:

- $d\hat{R}(k)$: the random F–S increment of the zero-mean stationary field $R'(x)$;
- $d\hat{A}(k)$: the random F–S increment of the zero-mean stationary field $A'(x)$; and
- $d\hat{B}(k)$: the random F–S increment of the composite random field (zero-mean and stationary):

$$B'(x) = \frac{\partial A'}{\partial x} - jk_0A'(x), \tag{27a}$$

$$\begin{aligned}\Rightarrow d\hat{B}(k) &= jkd\hat{A}(k) - jk_0d\hat{A}(k) \\ &= j(k-k_0)d\hat{A}(k).\end{aligned} \tag{27b}$$

Thus, applying the property of Eq. (B-3) yields Eqs. (27a), (27b), (28a) and (28b)

$$d\hat{f}_R(k) = -\omega^2U_0(\omega)d\hat{R}(k+k_0); \tag{28a}$$

$$\begin{aligned}d\hat{f}_A(k) &= jk_0V_0^2U_0(\omega)d\hat{B}(k+k_0) \\ &= -U_0(\omega)V_0^2k_0kd\hat{A}(k+k_0) \\ &= -\omega^2U_0(\omega)\frac{k}{k_0}d\hat{A}(k+k_0).\end{aligned} \tag{28b}$$

with $k_0(\omega) = k_0 = \omega/V_0$.

3.4. Inserting the Green's function and source terms to obtain closed form expression of solution in wavenumber-frequency space (first order random displacement field)

Having expressed the (stochastic) displacement solution rigorously in terms of random

Fourier–Stieltjes increments of the material properties (and in terms of the incident plane wave), we can now collect results to construct the explicit solution. This will be done by

(i) inserting the Green's function (Fourier space version) $G(k)$.

(ii) combining the u_R and u_A solutions, which correspond to $\rho(x)$ variable and $\alpha(x)$ variable, respectively [recall: $u = u_0 + \sigma_R u_R + \sigma_A u_A$].

Therefore, inserting the Fourier-space 1D Green's function (Eq. (A-8b)) in Eq. (B-12) and then substituting the previously obtained expression for Eqs. (28a) and (28b), yields

u_R:

$$\mathrm{d}\hat{u}_R(k) = 2\pi G(k)\mathrm{d}\hat{f}_R(k),$$

$$\mathrm{d}\hat{u}_R(k) = -U_0(\omega)\frac{k_0^2}{k_0^2 - k^2}\mathrm{d}\hat{R}(k+k_0). \quad (29a)$$

u_A:

$$\mathrm{d}\hat{u}_A(k) = 2\pi G(k)\mathrm{d}\hat{f}_A(k) = -U_0(\omega)\frac{k_0^2}{k_0^2 - k^2}\frac{k}{k_0}\mathrm{d}\hat{A}(k+k_0), \quad (29b)$$

where $U_0(\omega)$ is related to the deterministic displacement $u_0(x,\omega)$ as in Eq. (B-6).

Combining Eqs. (29a) and (29b), we find that the random part ($u' = u - u_0$) of the displacement has the random Fourier–Stieltjes increment given by

$$\mathrm{d}\hat{u}'(k) = \sigma_R\mathrm{d}\hat{u}_R(k) + \sigma_A\mathrm{d}\hat{u}_A(k) = U_0(\omega)\frac{k_0^2}{k^2 - k_0^2}\left\{\sigma_R\mathrm{d}\hat{R}(k+k_0) + \sigma_A\frac{k}{k_0}\mathrm{d}\hat{A}(k+k_0)\right\}, \quad (30)$$

where $k_0 = k_0(\omega) = \omega/V_0$. But

$$u'(x,\omega) = u(x,\omega) - u_0(x,\omega) = \int_{k=-\infty}^{k=+\infty} \mathrm{e}^{jkx}\mathrm{d}\hat{u}(k,\omega). \quad (31)$$

Finally, for an incident wave comprising a whole range of frequencies (= case of broadband $u_0(\omega)$), the space-time formulation of the random part of the wave is

$$u'(x,t) = u(x,t) - u_0(x,t) = \int \mathrm{e}^{j\omega t}u'(x,\omega)\mathrm{d}\omega, \quad (32)$$

which, on inserting Eq. (31) becomes

$$u'(x,t) = u(x,t) - u_0(x,t) = \int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{j\omega t}\int_{k=-\infty}^{k=+\infty}\mathrm{e}^{jkx}\mathrm{d}\hat{u}(k,\omega)\mathrm{d}\omega, \quad (33)$$

where Eq. (30) may be inserted for $\mathrm{d}\hat{u}(k,\omega)$.

In summary, the stochastic displacement wave field is expressed in closed form by Eqs. (31)–(33) in terms of the incident deterministic plane wave [its frequency content $U_0(\omega)$, its velocity $V_0 = \sqrt{\overline{\alpha}_G/\overline{\rho}_G}$, and its wavenumber $k_0(\omega) = \omega/V_0$] and in terms of the random material properties [log-coefficients $R(x)$ and $A(x)$]. To obtain the random stress wave, we must apply Hooke's law [$\tau = \alpha(x)\varepsilon$ in the 1-D case] with random log-coefficients [$\alpha(x) = \overline{\alpha}_G \exp \times (\sigma_A A'(x))$ in the 1-D case].

Eqs. (31)–(33), taken together, show that the first order perturbation field $u'(x,\omega)$ can be expressed as the product of the deterministic right-going wave $u_0(x,\omega)$ (Eq. (B-6)), times a zero-mean. Thus, from Eqs. (31)–(33) we can write, using the change of variables of integration $k' = k + k_0$,

$$u'(x,\omega) = \int_{-\infty}^{+\infty}\mathrm{e}^{j(k'-k_0)x}U_0(\omega)\frac{k_0^2}{(k'-k_0)^2 - k_0^2} \times \left\{\sigma_R\mathrm{d}\hat{R}(k') + \sigma_A\frac{k'-k_0}{k_0}\mathrm{d}\hat{A}(k')\right\}. \quad (34)$$

Recognizing that $U_0(x,\omega) = U_0(\omega)\mathrm{e}^{-jk_0x}$ (the far field solution in a homogeneous medium) we obtain

$$u'(x,\omega) = u_0(x,\omega)U'(x,\omega), \quad (35)$$

where $U'(x,\omega)$ is the random field with spectral representation

$$U'(x,\omega) = \int_{-\infty}^{+\infty} k_0^2 e^{jkx}\left[(k-k_0)^2 - k_0^2\right]^{-1} \times \left\{\sigma_R d\hat{R}(k) + \sigma_A\left(\frac{k}{k_0}-1\right)d\hat{A}(k)\right\}. \quad (36)$$

The ω-dependence is due to the relation $k_0 = \omega/V_0$.

Eq. (36) is an explicit representation of the random increments and standard deviation of the stiffness and density fluctuations of the random medium. To calculate the random wavefield, Eq. (36) is solved numerically (in Section 4) using the method of Shinozuka (1972). This method consists of constructing a model that is characterized as a stationary Gaussian process with zero mean, and a standard deviation equal to the square root of the spectral density of the density and stiffness fluctuations.

4. Numerical evaluation of stochastic wave fields

The general solution of the first order stochastic displacement field in the space-frequency domain is given by

$$u'(x,\omega) = u_0(x,\omega)U'(x,\omega), \quad (37)$$

where $u_0(x,\omega)$ is the deterministic right-going plane wave incoming from the far field and $U'(x,\omega)$ is the real-valued stationary random field related to random material properties described below. The deterministic displacement is

$$u_0(x,\omega) = u_0(\omega)e^{-jk_0x}, \quad (38)$$

where $k_0 = \omega/V_0$

$$V_0 = \left(\frac{\alpha_G}{\rho_G}\right)^{1/2},$$

in which ρ_G = the geometric mean density; and α_G = the geometric mean of the 1D stiffness coefficient α, that is $\alpha = \lambda + 2\mu$. In addition, the total stochastic displacement can be determined, to first order, by

$$u(x,\omega) = u_0(x,\omega) + u'(x,\omega), \quad (39)$$

or

$$u(x,\omega) = u_0(x,\omega)\left[1 + U'(x,\omega)\right].$$

The function $U'(x,\omega)$ is a frequency-dependent spatial random field that has the following properties: real-valued, zero-mean, and stationary (i.e., with a spectral representation). In addition, the function $U'(x,\omega)$ has the following Fourier–Stieltjes representation (using Eq. (34))

$$U'(x,\omega) = \int_{-\infty}^{+\infty} e^{jkx} d\hat{U}'(k,\omega), \quad (40)$$

where

$$d\hat{U}'(k,\omega) = \frac{k_0^2}{(k-k_0)^2 - k_0^2}\left\{\sigma_R d\hat{R}(k) + \sigma_A\left(\frac{k}{k_0}-1\right)d\hat{A}(k)\right\}. \quad (41)$$

In Eq. (41) the Fourier–Stieltjes increments correspond to the (normalized) Gaussian random fields $R = \ln\ \rho(x)$ and $A = \ln\ \alpha(x)$. The following R/A correlation model is used to simplify the formulation for U'.

We assume that $R(x)$ and $A(x)$ are stationary and correlated material properties. The proposed cross-correlation model among the two material properties $R(x)$ and $A(x)$ is to assume that the two-point cross-covariance tensor of $(R(x),\ A(x))$ is of the form

$$\begin{bmatrix} C_{RR}(\xi) & C_{RA}(\xi) \\ C_{AR}(\xi) & C_{AA}(\xi) \end{bmatrix} = \begin{bmatrix} 1 & \tilde{r} \\ \tilde{r} & 1 \end{bmatrix} C(\xi), \quad (42)$$

where $C(\xi)$ is a specified correlation common to both $R(x)$ and $A(x)$, $\tilde{r}$ is their cross-correlation coefficient (i.e., $-1 \le \tilde{r} \le +1$). Similarly, this implies that the cross-spectral density tensor has the special form

$$\begin{bmatrix} S_{RR}(k) & S_{RA}(k) \\ S_{AR}(k) & S_{AA}(k) \end{bmatrix} = \begin{bmatrix} 1 & \tilde{r} \\ \tilde{r} & 1 \end{bmatrix} S(k). \quad (43)$$

A pair of Gaussian random fields R and A having the above cross-correlation/cross-spectral structures have the form

$$R(x) = Y(x), \quad (44)$$

$$A(x) = \tilde{r}Y(x) + \sqrt{1-\tilde{r}^2}\,Z(x), \quad (45)$$

where $Y(x)$ and $Z(x)$ are independent normalized Gaussian fields, both having the same spatial correlation $C(\xi)$ and spectral density $S(k)$. In this case, we use the fact that $R(x)$ and $A(x)$ are normalized Gaussian fields. Alternatively, for non-normalized $R(x)$ and $A(x)$, more general relations are

$$R(x) = \bar{R} + \sigma_R Y(x) \quad (46a)$$

and

$$A(x) = \bar{A} + \sigma_A\left[\tilde{r}Y(x) + \sqrt{1-\tilde{r}^2}\,Z(x)\right], \quad (46b)$$

where $Y(x)$ and $Z(x)$ are as defined above.

Applying the Fourier–Stieltjes transform to Eqs. (46a) and (46b), we obtain the increments:

$$\mathrm{d}\hat{R}(k) = \mathrm{d}\hat{Y}(k) \quad (47a)$$

and

$$\mathrm{d}\hat{A}(k) = \tilde{r}\,\mathrm{d}\hat{Y}(k) + \sqrt{1-\tilde{r}^2}\,\mathrm{d}\hat{Z}(k). \quad (47b)$$

As a consequence, we can now rewrite the random $U'(x,\omega)$ given by Eqs. (40) and (41) in the following forms

$$U'(x,\omega) = U'_y(x,\omega) + U'_z(x,\omega) \quad (48)$$

where

$$U'_y(x,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{jkx}\,\mathrm{d}\hat{U}'_y(k,\omega),$$

in which

$$\mathrm{d}\hat{U}'_y = \frac{k_0^2}{(k-k_0)^2 - k_0^2} \times \left[\sigma_R + \sigma_A\left(\frac{k}{k_0} - 1\right)\tilde{r}\right]\mathrm{d}\hat{Y}(k).$$

In the same manner

$$U'_z(x,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{jkx}\,\mathrm{d}\hat{U}'_z(k,\omega) \quad (49)$$

where

$$\mathrm{d}\hat{U}'_z(k,\omega) = \frac{k_0^2}{(k-k_0)^2 - k_0^2} \times \left[\sigma_A\left(\frac{k}{k_0} - 1\right)\sqrt{1-\tilde{r}^2}\right]\mathrm{d}\hat{Z}(k),$$

or, alternatively and equivalently

$$U'(x,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{jkx}\,\mathrm{d}\hat{U}'(k,\omega), \quad (50)$$

where

$$\mathrm{d}\hat{U}'(k,\omega) = \frac{k_0^2}{(k-k_0)^2 - k_0^2} \times \left\{\left[\sigma_R + \sigma_A\left(\frac{k}{k_0} - 1\right)\tilde{r}\right]\mathrm{d}\hat{Y}(k) + \left[\sigma_A\left(\frac{k}{k_0} - 1\right)\sqrt{1-\tilde{r}^2}\right]\mathrm{d}\hat{Z}(k)\right\}. \quad (51)$$

In these equations, ω-dependence enters through $k_0 = \omega / V_0$. Also, the functions $\mathrm{d}\hat{Y}(k)$ and $\mathrm{d}\hat{Z}(k)$ are the random increments of independent random fields $Y(x)$ and $Z(x)$ having the same correlation/spectral structures. The latter is the common correlation structure $C(\xi)$ or spectral density $S(k)$ of all log-material properties after normalization.

To evaluate Eq. (51) numerically requires modification of this equation by introducing attenuation and expressing (Eq. (51)) in a more appropriate form for calculations. The Stieltjes random increments has the form

$$\mathrm{d}\hat{U}'(k,\omega) = \frac{1}{\left(\frac{k}{\tilde{k}_0} - 1\right)^2 - 1}\left\{\sigma_R\,\mathrm{d}\hat{R}(k) + \sigma_A\left(\frac{k}{\tilde{k}_0} - 1\right)\mathrm{d}\hat{A}(k)\right\} \quad (52)$$

Table 1
Gaussian model parameters

γ	D
0	$l/\sqrt{2\pi}$
1/2	0.4852256021
1	$l/2$
2	$l/\sqrt{2\pi}$

in which attenuation is introduced by expressing the wavenumber, k_0, as a complex variable given by

$$\tilde{k}_0 = \frac{\omega}{V_0}(1 - j\delta),$$

where $\delta = 0.5/Q$ and Q is the quality factor of the random medium.

Expressing $\mathrm{d}\hat{R}$ and $\mathrm{d}\hat{A}$ in terms of $\mathrm{d}\hat{Y}$ and $\mathrm{d}\hat{Z}$, we can write Eq. (52) as

$$\mathrm{d}\hat{U}'(k,\omega) = h(k,\omega)\mathrm{d}\hat{Y}(k) + g(k,\omega)\mathrm{d}\hat{Z}(k), \tag{53}$$

where

$$h(k,\omega) = \frac{1}{\left(\frac{k}{\tilde{k}_0} - 1\right)^2 - 1} \times \left[\sigma_R + \tilde{r}\sigma_A\left(\frac{k}{\tilde{k}_0} - 1\right)\right], \tag{54a}$$

and

$$g(k,\omega) = \frac{1}{\left(\frac{k}{\tilde{k}_0} - 1\right)^2 - 1}\sqrt{1 - \tilde{r}^2}\,\sigma_A \times \left(\frac{k}{\tilde{k}_0} - 1\right). \tag{54b}$$

Next, we apply the method of Shinozuka (1972, 1987) by replacing the random increments by

$$\mathrm{d}\hat{Y}(k) = \sqrt{S(k)\mathrm{d}k}\,\mathrm{e}^{j\phi(k)} \tag{55a}$$

and

$$\mathrm{d}\hat{Z}(k) = \sqrt{S(k)\mathrm{d}k}\,\mathrm{e}^{j\Psi(k)}, \tag{55b}$$

where $\phi(k)$ and $\Psi(k)$ are uniformly distributed phases over $[0,2\pi]$ for any fixed $k \neq 0$ and $S(k)$ is the spectral density for the 1D material property coefficients (including the log-density and log-stiffness). Thus, by substituting the random increments given by Eqs. (55a) and (55b) into Eq. (53), the resulting expression may be replaced in Eq. (49) to obtain

$$U'(x,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{jkx}\sqrt{S(k)\mathrm{d}k}\left\{h(k,\omega)\mathrm{e}^{j\phi(k)} + g(k,\omega)\mathrm{e}^{j\Psi(k)}\right\}. \tag{56}$$

Thus, substituting Eq. (56) in Eq. (39) the total stochastic particle displacement is given by

$$u(x,\omega) = u_0(x,\omega) + \int_{-\infty}^{\infty} \mathrm{e}^{-j(k_0-k)x}u_0(\omega) \times \sqrt{S(k)\mathrm{d}k}\left\{h(k,\omega)\mathrm{e}^{j\phi(k)} + g(k,\omega)\mathrm{e}^{j\Psi(k)}\right\}. \tag{57}$$

The particle displacement in the time domain is obtained by applying FFT to Eq. (57). To calculate spectral responses and full waveform seismic waves using Eq. (56), we will use the following density functions.

4.1. Gaussian model

$$S(k) = D|kl|^{\gamma}\mathrm{e}^{-\frac{1}{2}(kl)^2},$$

where l is the correlation length. For each γ we have to find the values of D such that $S(k)$ is normalized by satisfying the relation

$$\int_0^{\infty} S(k)\mathrm{d}k = \frac{1}{2}.$$

We therefore find that γ and D are given by the values listed in Table 1.

Table 2
Gauss–Markov model parameters

γ	D
0	$2l/\pi$
1/2	$4l/\pi\sqrt{2}$
1	l
2	$2l/\pi$

4.2. Gauss–Markov model

$$S(k) = D|kl|^{\gamma}\left[1+(kl)^2\right]^{-2}.$$

The values of γ and D for this model are given in Table 2.

5. Numerical results

To test the random wavefield solution given in this paper, we constructed a model assuming a plane compressional wave propagating in the direction of a vertical borehole. The plane wave propagates parallel to the borehole axis with a compressional wave velocity of 2000 m/s in a medium having an average density of 2.7 g/cc and a quality factor $Q = 50$. The solution generates a frequency domain transfer function every 5 Hz from 0 to 5115 Hz. There are 11 ideal borehole receivers located at 1-m intervals from 1 to 11 m in the medium. The standard deviation of the fluctuations in the log of the stiffness and density is 0.1 for each property. The length scale (l) of the spatial variations in the medium is 0.5 m, corresponding to a characteristic scattering frequency $(\lambda = 4l)$ of 1000 Hz. Since we used a cross-correlation coefficient of the density and stiffness fluctuations equal to 0.5, each random medium property is generated from the inverse transform of the k-space random field given by

$$S(k)\left[e^{i\phi(k)} + e^{j\psi(k)}\right],$$

where $\phi(k)$ and $\Psi(k)$ are random phase angles, and $S(k) = Dk^2l^2e^{-k^2l^2}$. Different sets of random phase angles are used for each different material property. In practice, the symmetric random field is represented by discrete points from $kl = 0.0$ at 15.0 by 0.001 intervals. This small interval is necessary to resolve peaks in the integrand having a half-width on the order of $\omega l/(QV)$. For this reason, the Q cannot be too large.

Fig. 1 shows the density and velocity profiles of a random medium generated using the parameters given above. It is easily observed that there is roughly a 10% variation in the density and velocity, consistent with the 0.1 log variation cited above. In a non-random medium, there would be no such variations and both plots would be flat.

Figs. 2 and 3 show seismograms for each receiver at frequencies of 400 and 800 Hz,

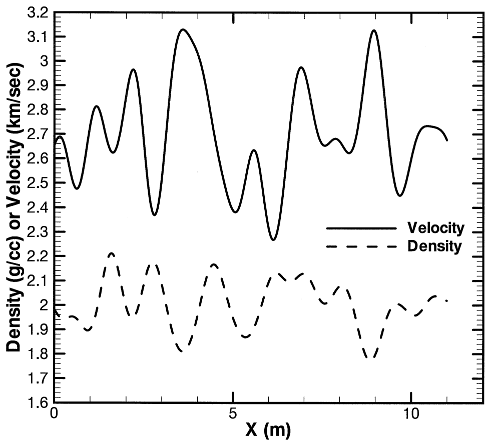

Fig. 1. Density and velocity variations of a random medium as a function of depth. Plots were generated using a correlation length $l = 0.5$ m corresponding to a characteristic scattering frequency of 1000 Hz.

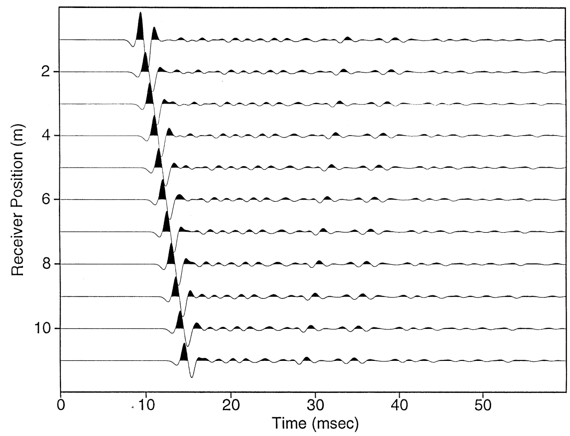

Fig. 2. Synthetic seismogram of a plane-wave propagating in a medium having the random rock physical properties depicted graphically in Fig. 1. The source center frequency used to produce this seismogram was 400 Hz.

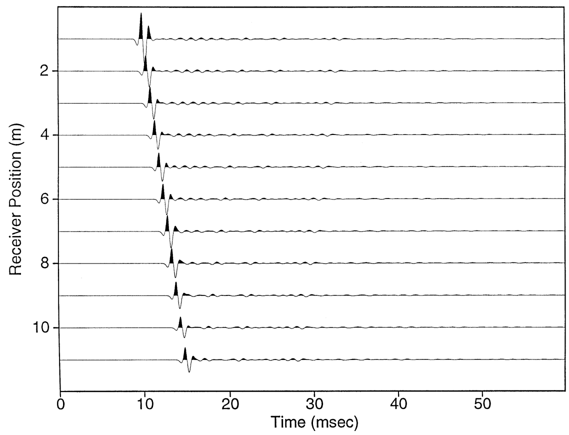

Fig. 3. Synthetic seismogram of a plane-wave propagating in a medium having the random rock physical properties depicted graphically in Fig. 1. The source center frequency used to produce this seismogram was 800 Hz.

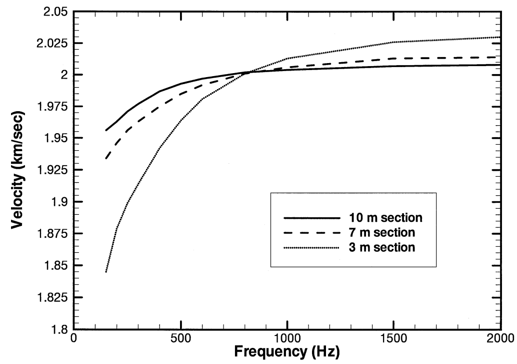

Fig. 4. Dispersion curves based on the time required to travel distances of 3, 7 and 10 m, respectively.

respectively. Note the slight amplitude and pulse width variations, although the value of $Q=50$ forces a general weakening of the pulse with distance at the higher frequencies. The amplitude and pulsewidth variations away from the main pulse would not be seen in a regular non-random medium. However, the appearance of these extra amplitude variations away from the main pulse occurs in actual field data. (See Parra et al. (1996).)

The method of calculating the dispersion curves is based on pulses and group velocities, not single frequencies and phase velocities. In our method, a Ricker wavelet of defined center frequency is convolved with the transfer function for each receiver in order to generate the seismogram. Arrival times are determined by locating a specified zero crossing in the time series with an automatic picking program. Fig. 4 shows the dispersion curve based on the time required to traverse distances of 3, 7 and 10 m. In fact, at the lower (dispersing) frequencies, the measured velocity varies with spatial location as well as frequency. That is, a different dispersion curve could be calculated between every pair of receivers. It is possible, then, to make a map of effective velocity as $V=V(x,\omega)$ using adjacent receivers to define the x values on the map (Fig. 5). This map shows that regions of positive $\mathrm{d}V/\mathrm{d}\omega$ are interspersed with regions of negative $\mathrm{d}V/\mathrm{d}\omega$. However, as dispersion is examined over longer distances (as in Fig. 4) these effects cancel, resulting in a net non-dispersive medium.

In Fig. 4 the shorter lengths should show more effects of the heterogeneities, while the longer lengths should be more representative of the medium as a whole. In the high frequency limit, all three dispersion curves approach the constant velocity of the ray-theoretical limit. This limit is nominally the average velocity of 2000 m/s, but will fluctuate according to the actual average velocity over the measurement interval. In the low-frequency limit, the disper-

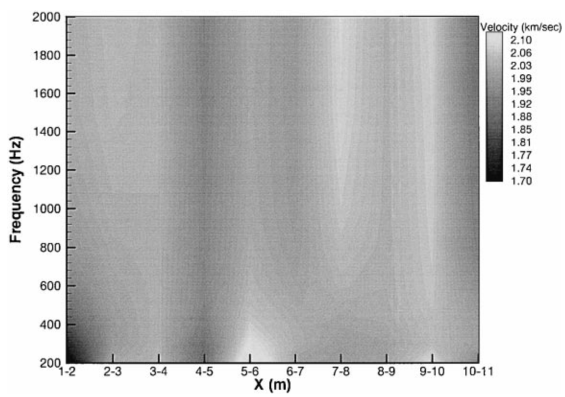

Fig. 5. Map of effective velocity showing the variations of velocity between pairs of receivers and frequency.

sion curve should also approach a bounding limit: the Backus, or effective medium, velocity. The two curves describing shorter path lengths do indeed appear to approach a limiting velocity. The longer path length curve could well approach a limit if lower frequencies were present. Unfortunately, the integral equation for the transfer function becomes very difficult to solve at low frequencies. This is due to a peak in the integrand with width proportional to $1/\omega$ and height proportional to ω. With more effort, this could be readily resolved by adaptive integration techniques. In the present case, however, the integral is also dependent on the phase angles ϕ_i and Ψ_i which must be distributed at fixed points in k-space and the peak must be well resolved by both the phase angles and the integration variable. Changing the location of the phase angles (or introducing new ones) would change the structure of the underlying medium, and so is not a viable option. We are thus limited to ''higher'' frequencies by the availability of computational resources.

6. Conclusions

A fundamental analysis of wave propagation in randomly heterogeneous medium was developed by treating the stochastic wave propagation problem in 1D space for plane-harmonic compressional seismic waves traveling in a medium having random material properties distributed in the vertical direction. The 1D random medium solution was developed in terms of standard deviations. The results demonstrate, in principle, that the solution can be used to simulate scattering attenuation caused by thin random-property layering in laminated reservoirs. The solution can be extended to include plane compressional waves incident at arbitrary angles and can be used to calculate scattering attenuation and velocity dispersion as a function of depth using the amplitude ratio method.

In addition, the theory presented in this paper has been extended to the development of the effective propagation vector of acoustic waves in randomly heterogeneous media. It is presented in a companion paper where an analytical solution is obtained in terms of the standard deviations of the density and the Young's modulus, as well as cross-correlation coefficients of the rock physical properties. Also, this companion paper introduces practical expressions for the attenuation and phase velocity of waves in random heterogeneous media that is applied to interpret interwell acoustic data recorded at the Buckhorn test site, Illinois.

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Appendix A. Fourier space–Green's function solution using ''informal'' fourier representations

We have seen in Eqs. (22) and (24b) that our perturbation equations for the elementary components of the random displacements [such as u_R due to $\rho(x)$, u_A due to $\alpha(x)\ldots$] were of the generic form, in space-time

$$\left(\frac{\partial^2}{\partial t^2} - V_0^2\frac{\partial^2}{\partial x^2}\right)u(x,\omega) = -f(x,t). \qquad \text{(A-1a)}$$

The space-frequency version of this expression is given by

$$\left(\omega^2 + V_0^2\frac{\partial^2}{\partial x^2}\right)u(x,\omega) = f(x,\omega). \qquad \text{(A-1b)}$$

Normally, we can carry the sequence of transforms one step further by transforming to wavenumber space, giving

$$\left(\omega^2 - V_0^2 k^2\right)\tilde{u}(k,\omega) = \tilde{f}(k,\omega), \tag{A-2}$$

and the informal solution

$$\tilde{u}(k,\omega) = \frac{\tilde{f}(k,\omega)}{\omega^2 - V_0^2 k^2}, \tag{A-3}$$

The informal Fourier representation is

$$u(x,\omega) = \int e^{jkx}\tilde{u}(k,\omega)\mathrm{d}k, \tag{A-4a}$$

$$\tilde{u}(k,\omega) = \frac{1}{2\pi}\int e^{-jkx}u(x,\omega)\mathrm{d}x, \tag{A-4b}$$

and similarly for $f(x,\omega)$ and $\bar{f}(k,\omega)$.

The previous calculations appear to be simple because (i) we are operating on a simple 1D model problem, and (ii) we have postponed for later a scrutiny of the postulated Fourier representations of the random (u) and (f) fields.

Regarding point (i), it will be instructive to re-derive the solution above by using a Green's function approach in Fourier space. This will be more analogous to the 3D case (where we have to use a dyadic Green's function).

Let us write the space-frequency Eq. (A-1b) in a generic way as

$$\mathscr{L}_0(u(x)) = f(x), \tag{A-5}$$

where the dependence on frequency (ω) is implicit. The Green's function for this problem is that which satisfies

$$\mathscr{L}_0(G(x,y))_y = \delta(y-x), \tag{A-6}$$

where the operator differentiates with respect to y (not x). Applying deterministic Fourier transforms to both sides yields,

$$\left(\omega^2 - V_0^2 k^2\right)\tilde{G}(k) = \frac{1}{2\pi}. \tag{A-7}$$

The Fourier space–Green's function is easily obtained from Eqs. (55a) and (55b) and is

$$\tilde{G}(k) = \frac{1}{2\pi}\frac{1}{\omega^2 - V_0^2 k^2}. \tag{A-8a}$$

This can also be represented as

$$\tilde{G}(k) = \frac{1}{2\pi}\frac{1}{\omega^2}\frac{k_0^2}{k_0^2 - k^2}, \quad k_0 = \frac{\omega}{V_0}. \tag{A-8b}$$

In space, the Green's function serves to express the solution as an integral over the source term. That is,

$$u(x) = \int G(x,y) f(y)\mathrm{d}y. \tag{A-9}$$

Since $G(x,y) = G(|y-x|)$, we can write the above integral as a convolution

$$u(x) = \int G(x-y) f(y)\mathrm{d}y. \tag{A-10}$$

Using the Fourier convolution theorem (related to Parseval's theorem) yields

$$\tilde{u}(k) = 2\pi\tilde{G}(k)\tilde{f}(k), \tag{A-11}$$

which is indeed equivalent to the solution already obtained (see Eq. (A-2)), provided that the Green's function given by Eq. (A-8a) is used.

Note: Eq. (A-11) is easily proved as follows:

$$\begin{aligned} U(x) &= \int_{-\infty}^{\infty} f(y)G(x-y)\mathrm{d}y \\ &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty}\tilde{G}(k)e^{+ik(x-y)}\mathrm{d}k\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\tilde{G}(k)\left[\int_{-\infty}^{\infty} f(y)e^{-iky}\mathrm{d}y\right]e^{ikx}\mathrm{d}k \\ &= \int_{-\infty}^{\infty}\tilde{G}(k)\left(2\pi\tilde{f}(k)\right)e^{ikx}\mathrm{d}k \end{aligned}$$

so that

$$\tilde{u}(k) = 2\pi\tilde{f}(k)\tilde{G}(k). \qquad \text{Q.E.D.}$$

Appendix B. Expression of stochastic solution in terms of random Fourier–Stieltjes increments

The difficulty that has been eluded so far is that the Fourier decompositions used in Appendix A are not usually allowed, strictly speaking, for random measure (Fourier–Stieltjes measure). Instead of Fourier components such as

$\tilde{u}(k)$, one usually uses only random Fourier–Stieltjes increments. Only in the case of exceptionally smooth random fields can we write equivalently for $\mathrm{d}\hat{u}(k)$. Moreover, for a random field, we employ a Fourier representation that is orthogonal in a statistical sense. Only for a zero-mean stationary random field are we guaranteed that the random increments are uncorrelated for different wavenumbers. We can also encounter the case where two random fields which are separately stationary are not necessarily jointly stationary, i.e., they are not ''stationarily correlated''. All of these different aspects have to be treated carefully in order to obtain a tractable characterization of the stochastic wavefield. We begin with a re-formulation of the Fourier space representation.

B.1. Fourier–Stieltjes representations

A priori, we know only that the random log-coefficients $A'(x)$ and $R'(x)$ are zero-mean stationary random fields (and they are either uncorrelated, or stationarily correlated, as per our assumptions). We are not sure (a priori) that this holds also for the solution $u(x,t)$ or $u(x,\omega)$ or the random source term $f(x,t)$ or $f(x,\omega)$. With regard to the source term, Eqs. (23) and (25) indicate that the random source term ''f'' is of the generic form [in (x,t) space]

$$f(x,t) = a'(x) u_0(x,t) \tag{B-1a}$$

or [in (x,ω) space]

$$f(x,\omega) = a'(x) u_0(x,\omega), \tag{B-1b}$$

where $a'(x)$ is a zero-mean Gaussian random field and $u_0(x,\omega)$ is a (deterministic) planewave. Owing to the stationarity of $a'(x)$, and using the generalized Fourier–Stieltjes integrals mentioned above, it can be shown that $f(x,\omega)$ is a zero-mean random field with the following Fourier representation

$$f(x,\omega) = \int_{\Re} \mathrm{e}^{jkx} \mathrm{d}\hat{f}(k,\omega). \tag{B-2}$$

The Fourier increment is given by (omitting dependence on ω)

$$\mathrm{d}\hat{f}(k) = \int_{k' \in \Re} \mathrm{d}\hat{u}_0(k-k') \mathrm{d}\hat{a}(k'), \tag{B-3}$$

where this result may be derived rigorously in generic notation as follows:

Let

$$F(x) = \alpha(x) A(x)$$

with

$$\alpha(x) = \int_{\Re} \mathrm{e}^{jkx} \tilde{\alpha}(k) \mathrm{d}k = \int_{\Re} \mathrm{e}^{jkx} \mathrm{d}\hat{\alpha}(k),$$

and

$$A(x) = \int_{\Re} \mathrm{e}^{jkx} \mathrm{d}\hat{A}(k),$$

$$\Rightarrow F(x) = A(x) \alpha(x)$$

$$= \int \mathrm{e}^{jkx} \mathrm{d}\hat{A}(k) \int \mathrm{e}^{jk'x} \mathrm{d}\hat{\alpha}(k)$$

$$\Rightarrow F(x) = \int_{k \in \Re} \mathrm{d}A(k) \int_{k' \in \Re} \mathrm{e}^{j(k+k')x} \mathrm{d}\hat{\alpha}(k').$$

Let

$$K = k + k' \in \Re \Rightarrow$$

$$F(x) = \int_{k \in \Re} \mathrm{d}A(k) \int_{K \in \Re} \mathrm{e}^{jKx} \mathrm{d}\hat{\alpha}(K-k).$$

Finally, changing the order of integration without changing the result

$$F(x) = \int_{K \in \Re} \mathrm{e}^{jKx} \left(\int_{k \in \Re} \mathrm{d}\hat{A}(k) \mathrm{d}\alpha(K-k) \right).$$

But, this latter expression is precisely of the form of a Fourier–Stieltjes spectral representation. That is

$$F(x) = \int_{K \in \Re} \mathrm{e}^{jKx} \mathrm{d}\hat{F}(K),$$

with a random Fourier–Stieltjes increment of the form

$$\mathrm{d}\hat{F}(k) = \int_{k \in \Re} \mathrm{d}\hat{A}(k) \mathrm{d}\hat{\alpha}(K-k).$$

Similarly, the product

$$\langle \mathrm{d}\hat{f}(k)\mathrm{d}\hat{f}^*(k')\rangle = \int_{\Re}\mathrm{d}\hat{u}_0(k-w)\mathrm{d}\hat{u}_0^*(k'-w)S_{aa}(w)\mathrm{d}w, \qquad \text{(B-4)}$$

where $S_{aa}(k)$ is the spectral density function of the random field $a'(x)$.

Proof. From Eq. (B-3)

$$\mathrm{d}\hat{f}(k)\mathrm{d}\hat{f}^*(k') = \int \mathrm{d}\hat{u}_0(k-v)\mathrm{d}\hat{a}(v)\mathrm{d}\hat{u}_0^*(k'-w)\mathrm{d}\hat{a}^*(w) = \int \mathrm{d}\hat{u}_0(k-v)\mathrm{d}\hat{u}_0^*(k'-w) \times \mathrm{d}\hat{a}(v)\mathrm{d}\hat{a}^*(w).$$

Assuming ''stationarily correlated zero-mean random fields'', we have, by definition of such fields, the relationship that

$$\langle \mathrm{d}\hat{a}(v)\mathrm{d}\hat{a}^*(w)\rangle = S_{aa}(v)\delta(v-w)\mathrm{d}v\mathrm{d}w. \qquad \text{(B-5)}$$

This then yields

$$\langle \mathrm{d}\hat{f}(k)\mathrm{d}\hat{f}^*(k')\rangle = \int \mathrm{d}\hat{u}_0(k-v)\mathrm{d}\hat{u}_0^*(k'-w) \times \langle \mathrm{d}\hat{a}(v)\mathrm{d}\hat{a}^*(w)\rangle = \int \mathrm{d}\hat{u}_0(k-w)\mathrm{d}\hat{u}_0^*(k'-w)S_{aa}(w)\mathrm{d}w.$$

Q.E.D.

The spectral density function $S_{aa}(k)$ is, therefore, defined through the relationship given by Eq. (B-5), by definition of what is meant by a stationarily correlated zero-mean random field.

In Eq. (B-4), we note that, for a given frequency, ω, the plane-wave $u_0(x,\omega)$ must be single-spiked in wavenumber space. That is, $u_0(x,\omega)$ will vanish except for a particular wavenumber which we may denote by k_0 (see Eq. (A-8a)). As a consequence, we will have that, for a plane wave, except for $k=k'$. This is sufficient to conclude that: If $u_0(x,\omega)$ represents a deterministic plane-wave (with single-spiked wavenumber spectrum), and if $a'(x)$ is a zero-mean stationary random field, then $f(x,\omega) = a'(x)u_0(x,\omega)$ is also a zero-mean stationary random field.

Specifically, for a right-going plane-wave, the deterministic solution is given in frequency space by

$$u_0(x,\omega) = U_0(\omega)\mathrm{e}^{-jk_0x} \quad (k_0=\omega/V_0). \qquad \text{(B-6)}$$

In wavenumber space, this yields a Fourier-increment

$$\mathrm{d}\hat{u}_0(k') = U_0(\omega)\delta(k'+k_0)\mathrm{d}k'. \qquad \text{(B-7)}$$

When this is inserted in Eq. (B-3), the result is

$$\mathrm{d}\hat{f}_\omega(k) = U_0(\omega)\mathrm{d}\hat{a}(k+k_0), \qquad \text{(B-8)}$$

and similarly when Eq. (B-7) is substituted into Eq. (B-4), the result is

$$\langle \mathrm{d}\hat{f}_\omega(k)\mathrm{d}\hat{f}_\omega^*(k')\rangle = |U_0(\omega)|^2 S_{aa}(k+k_0) \times \delta(k'-k)\mathrm{d}k\mathrm{d}k'. \qquad \text{(B-9)}$$

Proof.

$$\langle \mathrm{d}\hat{f}(k)\mathrm{d}\hat{f}^*(k')\rangle = \int \mathrm{d}\hat{u}_0(k-w)\mathrm{d}\hat{u}_0^*(k'-w)S_{aa}(w)\mathrm{d}w = |U_0(w)|^2\int\delta(k-w+k_0) \times \mathrm{d}k\,\delta(k'-w+k_0)\mathrm{d}k' S_{aa}(w)\mathrm{d}w = |U_0(w)|^2 S_{aa}(k+k_0)\delta(k'-k)\mathrm{d}k\mathrm{d}k'.$$

Note that $k_0 = k_0(\omega)$. (A better notation for ''k_0'' may be ''k_ω''.) Also note that Eq. (B-9) expresses the product $\langle \mathrm{d}f_\omega(k)\mathrm{d}f_\omega^*(k')\rangle$ for the special case $\omega'=\omega$. Since this product always appears as an integrand (in wavenumber integrals), then applying the usual rules of integration with delta-functions shows that Eq. (B-9) is equivalent to

$$\langle \mathrm{d}\hat{f}(k)\mathrm{d}\hat{f}^*(k')\rangle = \begin{cases} 0 & \text{if } k\neq k', \\ |U_0(\omega)|^2 S_{aa}(k+k_0)\mathrm{d}k & \text{if } k=k'. \end{cases} \qquad \text{(B-10)}$$

where we dropped the subscript ω for convenience.

From Eq. (B-5), the results in Eqs. (B-9) and (B-10) demonstrate that the source term

$f(x,\omega)$ (in frequency-space)

is a zero-mean stationary random function of x, with spectral density function

$$S_{ff}(k) = |U_0(\omega)|^2 S_{aa}(k+k_0), \qquad \text{(B-11)}$$

and with random increment given by Eq. (B-8). Here, recall that

- $|U_0(\omega)|$ is the modulus or amplitude of the (complex) plane wave input at frequency ω.
- $S_{aa}(k)$ represents the spectral density of the random material property $a'(x)$.
- $k_0 = \omega/V_0$ is a frequency-dependent wavenumber characteristic of the incident plane wave (V_0 being a constant material property).
- The plane-wave was assumed to be right-going, i.e., along positive x. This corresponds to a wavenumber having a minus sign.

The Fourier space–Green's function solutions in Appendix A can now be expressed more correctly in terms of Fourier–Stieltjes increments as follows (see Eq. (A-11))

$$\mathrm{d}\hat{u}(k) = 2\pi \tilde{G}(k) \mathrm{d}\hat{f}(k), \qquad \text{(B-12)}$$

where all quantities depend on frequency ω. The Green's function in Fourier space was given by Eq. Eq. (A-8a). For $\mathrm{d}\hat{f}(k)$, we have shown above that, if $f(x)$ is of the form

$$f(x,\omega) = a'(x) u_0(x,\omega),$$

then is given by Eq. (B-8). That is,

$$\mathrm{d}\hat{f}(k) = U_0(\omega) \mathrm{d}\hat{a}(k+k_0).$$

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